Technical report #2 (in System and Control)

presented to

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Introduction

There are a few things needed to be mentioned about this Technical Report #2. Firstly, the last section of this report contains what was to be included in what would have been Technical Report #1 Version 2. This means that there will no longer be a new version for Technical Report #1, and for other technical reports as well. Instead, the next technical reports will built upon their predecessors, as well as upon other things.
By doing it this way I hope that the organization of the reports will become even simpler. The idea is to link research works and studies together so that they may be easier to find when needed.

Second order singularly perturbed sytem

2^{nd} **May 1998**
Consider the system described by a differential equation

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} = u, \tag{1}$$

or written as state equation

$$\dot{x}_1 = x_2 \tag{2}$$
$$\varepsilon \dot{x}_2 = -x_2 + u. \tag{3}$$

Some knowledge about the system

By letting the right-hand side of the state equation be zero, every point on the x_1-axis is an equilibrium point.
Equation 2 and 3 represent a model in *standard form* [Kha96] since Equation 3 has a real isolated root when $\varepsilon = 0$.
(A model

$$\dot{x} = f(t, x, z, \varepsilon), \quad x \in \mathcal{R}^n \tag{4}$$
$$\varepsilon \dot{z} = g(t, x, z, \varepsilon), \quad z \in \mathcal{R}^n \tag{5}$$

is said to be in *standard form* if and only if [Kha96]

$$0 = g(t, x, z, 0)$$

has k isolated real roots

$$z = h_i(t, x), \quad i = 1, 2, \ldots k.$$

And the model in Equation 4 and 5 reduces to a *quasi-steady-state model* or a *slow model*

$$\dot{x} = f(t, x, h(t, x), o). \tag{6}$$

)
From the Jacobian matrix

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=(x_1,0)} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\varepsilon} \end{bmatrix}$$

the Eigenvalues are $0, -\frac{1}{\varepsilon}$. Thus the system has an equilibrium subspace and the qualitative behaviour of the trajectories is summarized in Table 1.
From Table 1, the system is stable when $\varepsilon > 0$ and unstable when $\varepsilon < 0$.

Response with various values of ε

Time responses when $u = 0$ of the state variables of this system were shown in Figure 1 (a) to (f), and the state plane were shown in Figure 2 (a) to (f). Figure 1 and 2 shows that the system is stable when $\varepsilon > 0$ and unstable when $\varepsilon < 0$. When $|\varepsilon|$ becomes smaller the response of the system becomes more rapid as

$$\dot{x}_2 = \frac{1}{\varepsilon} x_2$$

Values of ε	Nature of trajectories
$\varepsilon > 0$	All trajectories converge to the equilibrium subspace
$\varepsilon = 0$	Degenerates, reduced to a 1^{st}-order system
$\varepsilon < 0$	All trajectories diverge away from the equilibrium subspace

Table 1: *Qualitative behaviour of the trajectory.*

rapidly coverges to its root $x_2 = 0$ when $\varepsilon \to 0$.
The closer ε gets to zero the more parallel the trajectory becomes to the x_2-axis. Equation 1 when $\varepsilon \to 0$ and $u = 0$ would become

$$\dot{x}_1 = 0, \quad \text{or} \quad x_2 = 0,$$

which has the solution

$$x_1(t) = x_1(0), \quad \text{and} \quad x_2(t) = 0.$$

The singular point $\varepsilon = 0$ is the point of discontinuity because the solution is stable when $\varepsilon \to 0^+$ and unstable when $\varepsilon \to 0^-$.

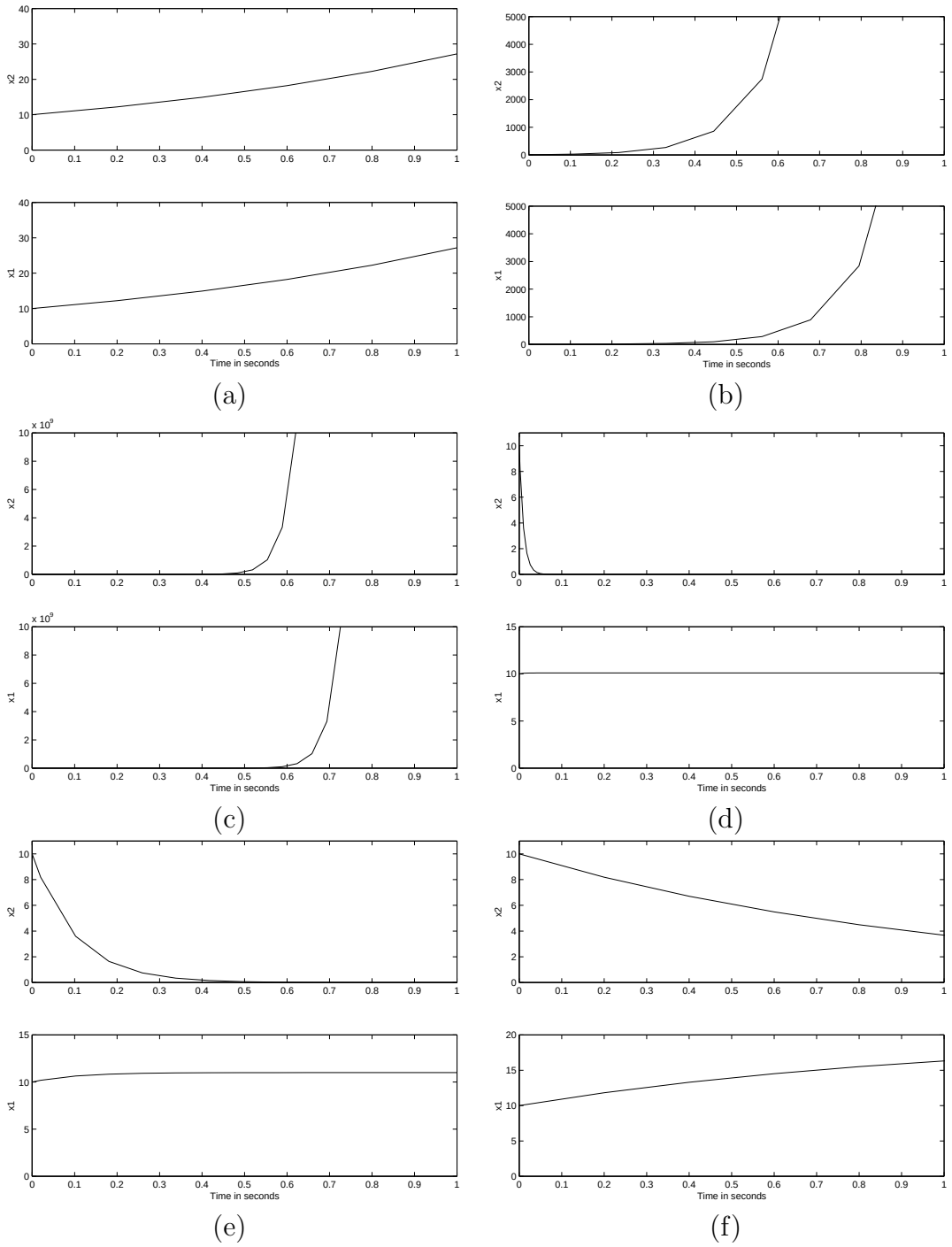


Figure 1: *Time responses of state variables. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*.*

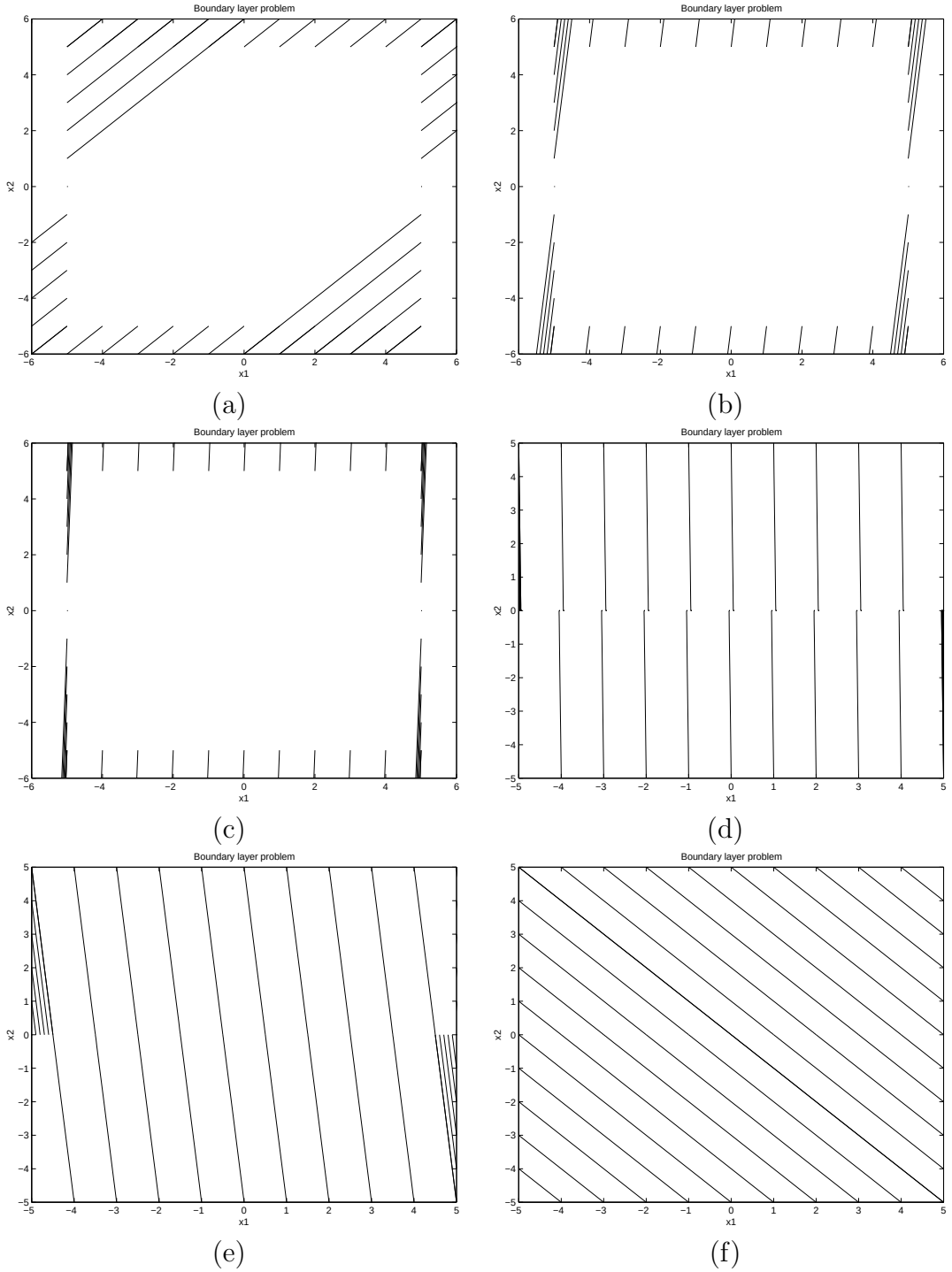


Figure 2: *State planes. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*.*

The phase planes for various ε are shown again in Figure 3, this time with the Initial Condition Set 1 as the initial conditions. When ε gets smaller the slope of the trajectory becomes steeper as the rate in which x_2 approaches its root becomes faster.

Initial Condition Set 1 *The set of n points described by*

```
n = 0
for i = -5, -3,..., 5
     x1(0) = i, x2(0) = i
     x1(0) = i, x2(0) = -i
     n = n+2
endfor
```

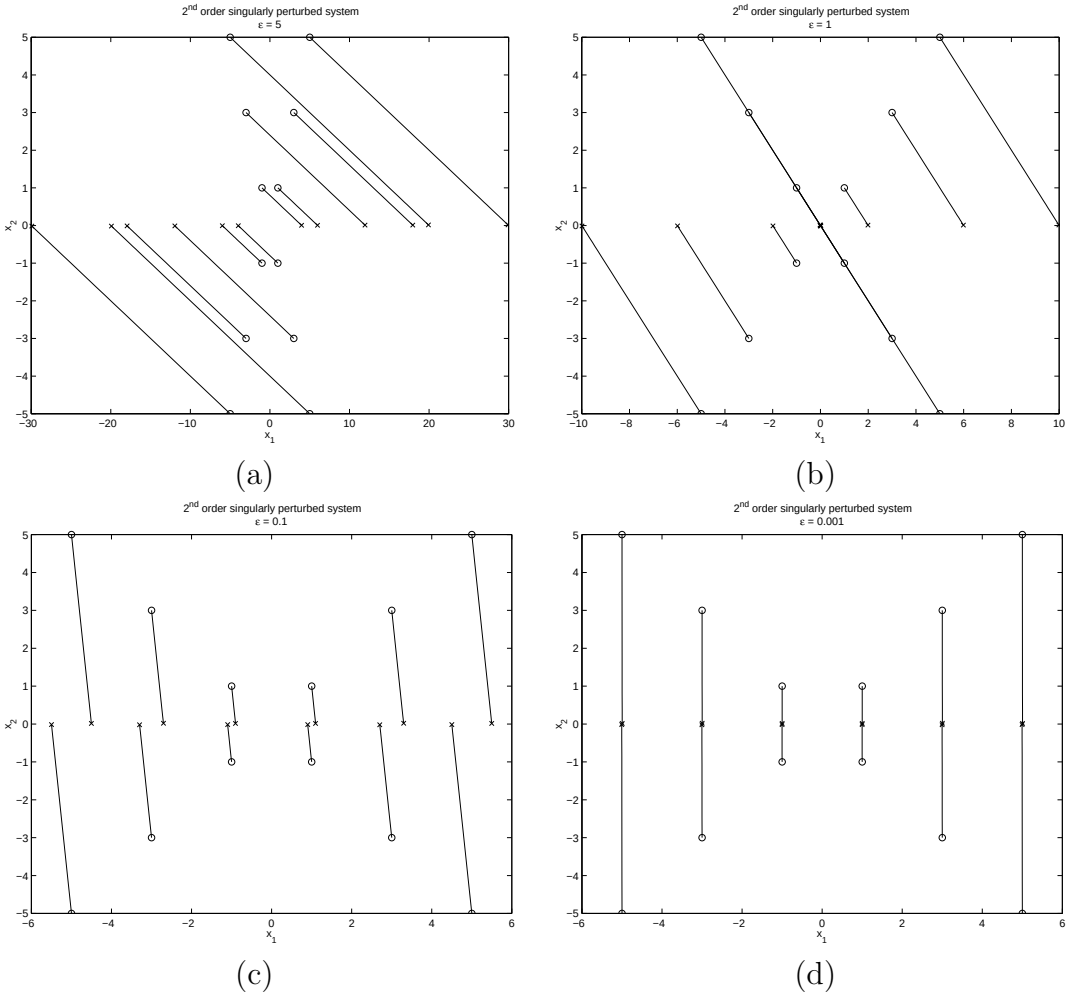


Figure 3: *State planes (a)* $\varepsilon = 5$*, (b)* $\varepsilon = 1$*, (c)* $\varepsilon = 0.1$*, (d)* $\varepsilon = 0.001$

With feedback of a sign function

Feed the sign function of the states

$$u = \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$$

back into the input and the results was obtained in Figure 4 (a) to (f) and in Figure 5 (a) to (f). (Though this may not be a good choice of input comparing with, for example $u = K\,\mathrm{sgn}\,(Cx_1 + x_2)$, when $C, K \in \mathcal{R}$.) Figure 4 shows that the response of the controlled system changes more rapidly when $|\varepsilon|$ becomes smaller. The system is stable when $\varepsilon > 0$ and unstable when $\varepsilon < 0$. Both Figure 1 in the previous section and 4 (d) to (f) show the boundary-layer characteristic of the system. In both figures the boundary-layer is $t \approx [0, 0.05]$, $[0, 0.4]$, and $[0, 1]$ when $\varepsilon = 0.01$, 0.1, and 1 respectively. The increase in speed of the response as ε decrease in magnitude applies to the case when $\varepsilon < 0$ as well as when $\varepsilon > 0$. In other words, the response becomes faster as $|\varepsilon| \to 0$.
The phase planes when the system is unstable (Figure 5 (a) to (c)) show only a small part of the trajectory due to the fact that the trajectory goes very far away from the origin. In all intent and purpose they only show that the system is unstable and the trajectory runs away.
With the control as shown in Equation the Equation 2 and 3 become

$$\begin{aligned} \dot{x}_1 &= x_2 && (7)\\ \varepsilon\dot{x}_2 &= -x_2 + \mathrm{sgn}\,(x_1 + x_2)\,. && (8) \end{aligned}$$

The root of Equation 8 when $\varepsilon = 0$

$$0 = -x_2 + \mathrm{sgn}\,(x_1 + x_2)$$

is

$$\bar{x}_2 = \mathrm{sgn}\,(\bar{x}_1 + \bar{x}_2)$$

which can not be written in a form

$$\bar{x}_2 = h(t, \bar{x}_1). \tag{9}$$

In other words, Equation 9 has no isolated root. Therefore one can not isolate the fast mode from the slow mode by introducing a new variable

$$y = x_2 - h\,(t, x_1)\,.$$

But for the purpose of finding the boundary-layer model, let

$$h\,(t, x_1, x_2) = \mathrm{sgn}\,(x_1 + x_2)\,. \tag{10}$$

Substitute x_2 in Equation 8 with x_2 obtained from

$$\begin{aligned} y &= x_2 - h\,(t, x_1, x_2)\\ &= x_2 - \mathrm{sgn}\,(x_1 + x_2)\\ x_2 &= y + \mathrm{sgn}\,(x_1 + x_2)\,. \end{aligned}$$

And introduce a new time variable τ obtained by

$$\begin{aligned} \varepsilon\frac{dy}{dt} &= \frac{dy}{d\tau} && (11)\\ \frac{d\tau}{dt} &= \frac{1}{\varepsilon} && (12)\\ \tau &= \tau_0 + \int_{t_0}^{t} t\,dt\\ &= \frac{t - t_0}{\varepsilon}, \quad \tau_0 = 0. && (13) \end{aligned}$$

The new time variable τ is generally known as a *stretched* time variable.
Then

$$\varepsilon\left[\frac{dy}{dt}+\frac{d}{dt}\text{sgn }(x_1+x_2)\right] = -y \tag{14}$$

$$\frac{dy}{d\tau}+\frac{d}{d\tau}\text{sgn }(x_1+x_2) = -y \tag{15}$$

$$\frac{dy}{d\tau}+\frac{d}{d\tau}\text{sgn }(x_1+y+\text{sgn }(x_1+x_2)) = -y \tag{16}$$

$$\frac{dy}{d\tau} = -y-\frac{d}{d\tau}\text{sgn }(x_1+y+\text{sgn }(x_1+x_2)) \tag{17}$$

Therefore the boundary-layer model can be represented by either one of the following,

$$\frac{dy}{d\tau}=\begin{cases} -y-\frac{d}{d\tau}\text{sgn }(x_1+y+1), & x_1+x_2>0 \\ -y-\frac{d}{d\tau}\text{sgn }(x_1+y), & x_1+x_2=0 \\ -y-\frac{d}{d\tau}\text{sgn }(x_1+y-1), & x_1+x_2<0 \end{cases}. \tag{18}$$

To find the reduced model look firstly at the three possible values which x_2 may take

$$\bar{x}_2=\begin{cases} +1, & \bar{x}_1+\bar{x}_2>0 \\ 0, & \bar{x}_1+\bar{x}_2=0 \\ -1 & \bar{x}_1+\bar{x}_2<0 \end{cases}. \tag{19}$$

Then the reduced problem becomes

$$\dot{x}_1=\begin{cases} +1 \\ 0 \\ -1 \end{cases} \tag{20}$$

with these three roots respectively.
The discontinuous nature of the sign function makes the analysis of the variable structure control system difficult. This is due to the fact that one can not analytically find

$$\frac{\partial \text{sgn}\, x(t)}{\partial t}, \quad \frac{\partial \text{sgn}\, x(t)}{\partial x_i}, \quad i=1,2,\ldots,n,$$

where $x(t)$ is an n-tuple vector.
One way to overcome or go around this problem is to use a numerical approximation for the sign function where necessary.

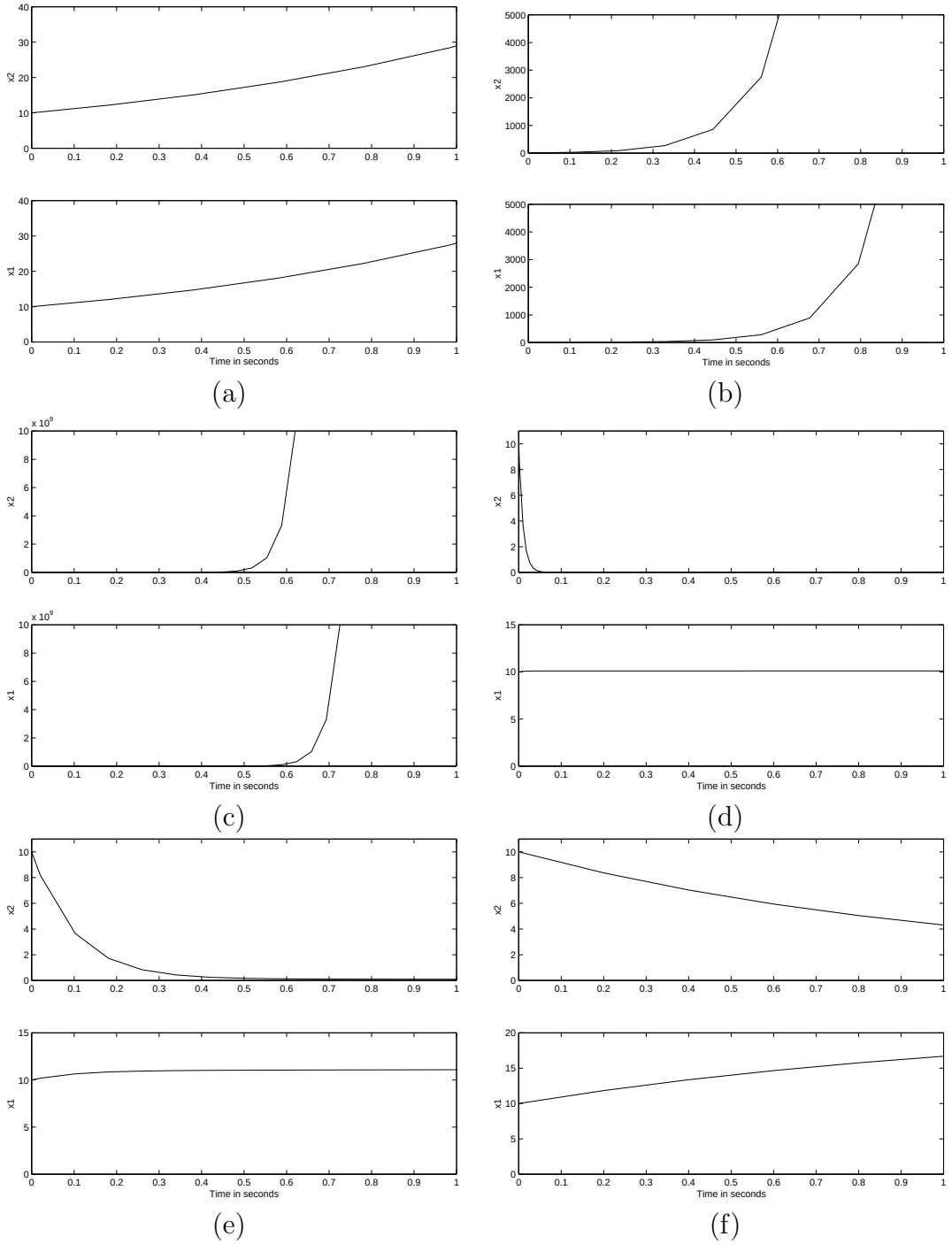


Figure 4: *Time responses of state variables. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*.*

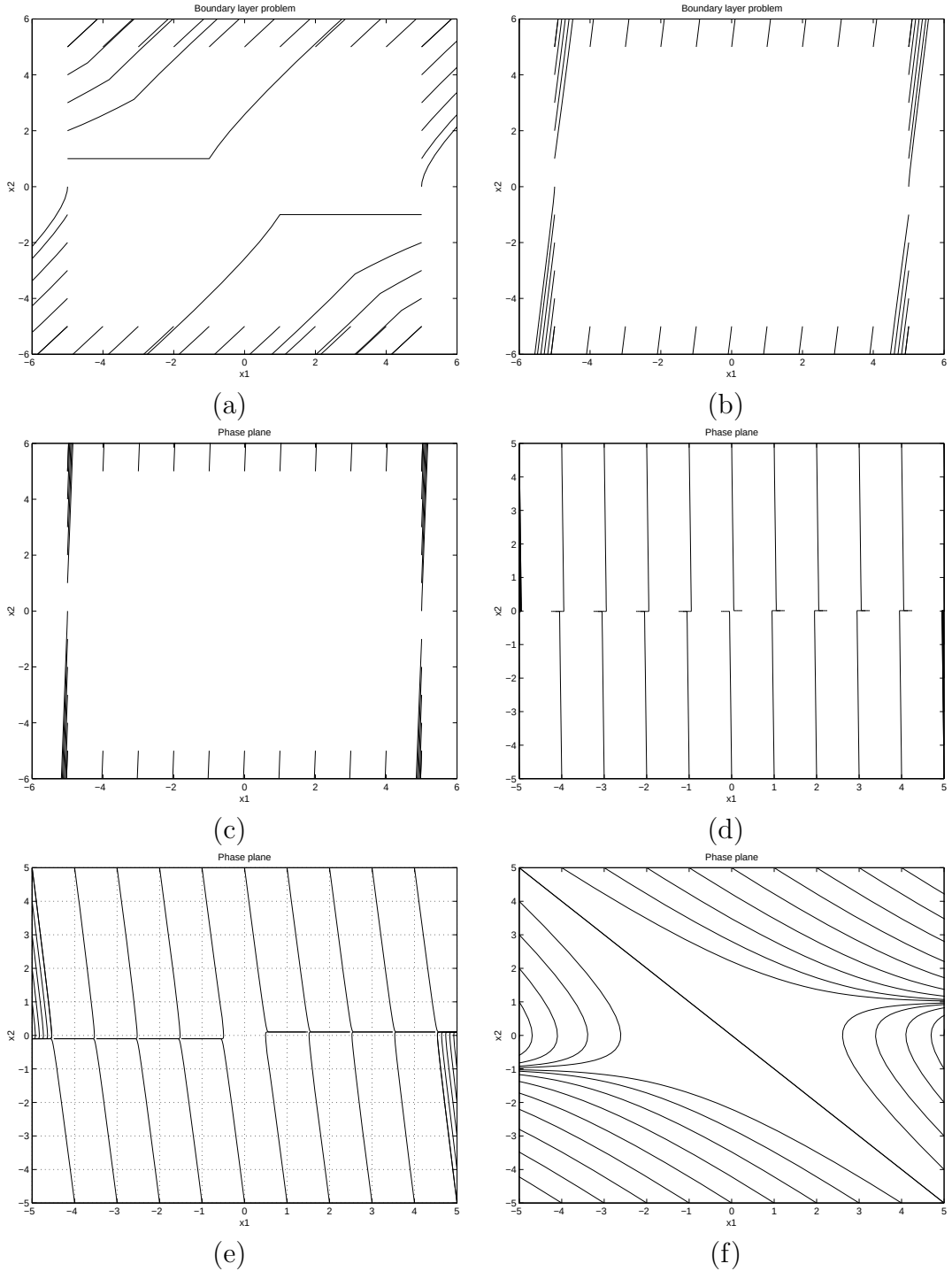


Figure 5: *State planes. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*.*

With feedbacks of a sign function and state variables

Feed the states as well as the sign function of the states back into the input

$$u = -(x_1 + x_2) + \text{sgn}\,(x_1 + x_2) \tag{21}$$

and the results was obtained in Figure 6 (a) to (f) (state variables) and Figure 7 (a) to (f) (phase planes). Similar to the previous studies, the system is unstable for $\varepsilon < 0$ and stable for $\varepsilon > 0$. Figure 6 (d) to (f) ($\varepsilon > 0$) shows that the system has a two-timescale property which is the characteristic of a singularly perturbed system. Having a fast response within the boundary-layer and a slow response elsewhere, the boundary-layer becomes narrower and the response faster as $\varepsilon \to 0^+$. Figure 6 (d) to (f)shows that the boundary-layer is $t \approx [0, 0.05]$, $[0, 0.23]$, and $[0, 0.54]$ when $\varepsilon = 0.01$, 0.1, and 1 respectively.
The control input of Equation 21 turns the system (Equation 2 and 3) into

$$\begin{aligned} \dot{x}_1 &= x_2 & (22)\\ \varepsilon\dot{x}_2 &= -x_1 - 2x_2 + \text{sgn}\,(x_1 + x_2). & (23) \end{aligned}$$

Equation 23 when $\varepsilon = 0$ becomes

$$0 = -x_1 - 2x_2 + \text{sgn}\,(x_1 + x_2), \quad \text{or} \quad x_2 = -\frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\,(x_1 + x_2). \tag{24}$$

In order to find the boundary-layer model, change the variable x_2 to

$$y = x_2 - x_2|_{\varepsilon=0} = x_2 + \frac{1}{2}x_1 - \frac{1}{2}\text{sgn}\,(x_1 + x_2). \tag{25}$$

in other words

$$\begin{aligned} x_2 &= y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\,(x_1 + x_2) & (26)\\ &= y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\,(x_1 + y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\,(x_1 + x_2)), & (27) \end{aligned}$$

which is essentially

$$x_2 = \begin{cases} y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\left(y + \frac{1}{2}x_1 + \frac{1}{2}\right), & x_1 + x_2 > 0 \\ y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\left(y + \frac{1}{2}x_1\right), & x_1 + x_2 = 0 \\ y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\left(y + \frac{1}{2}x_1 - \frac{1}{2}\right), & x_1 + x_2 < 0 \end{cases}. \tag{28}$$

And use the stretched time variable τ in Equation 12.
Substitute Equation 28 into Equation 23 to get the boundary-layer model

$$\begin{aligned} \varepsilon\left[\frac{dy}{dt} - \frac{1}{2}\frac{dx_1}{dt} + \frac{1}{2}\text{sgn}\,(x_1 + x_2)\right] &= -2y \\ \varepsilon\left[\frac{dy}{dt} - \frac{y}{2} + \frac{1}{4}x_1 - \frac{1}{4}\text{sgn}\,(x_1 + x_2) + \frac{1}{2}\text{sgn}\,(x_1 + x_2)\right] &= -2y \\ \frac{dy}{d\tau} &= -\frac{3}{2}y - \frac{1}{4}x_1 - \frac{1}{4}\text{sgn}\,(x_1 + x_2), \end{aligned}$$

which is essentially

$$\frac{dy}{d\tau} = \begin{cases} -\frac{3}{2}y - \frac{1}{4}x_1 - \frac{1}{4}, & x_1 + x_2 > 0 \\ -\frac{3}{2}y - \frac{1}{4}x_1, & x_1 + x_2 = 0 \\ -\frac{3}{2}y - \frac{1}{4}x_1 + \frac{1}{4}, & x_1 + x_2 < 0 \end{cases}. \tag{29}$$

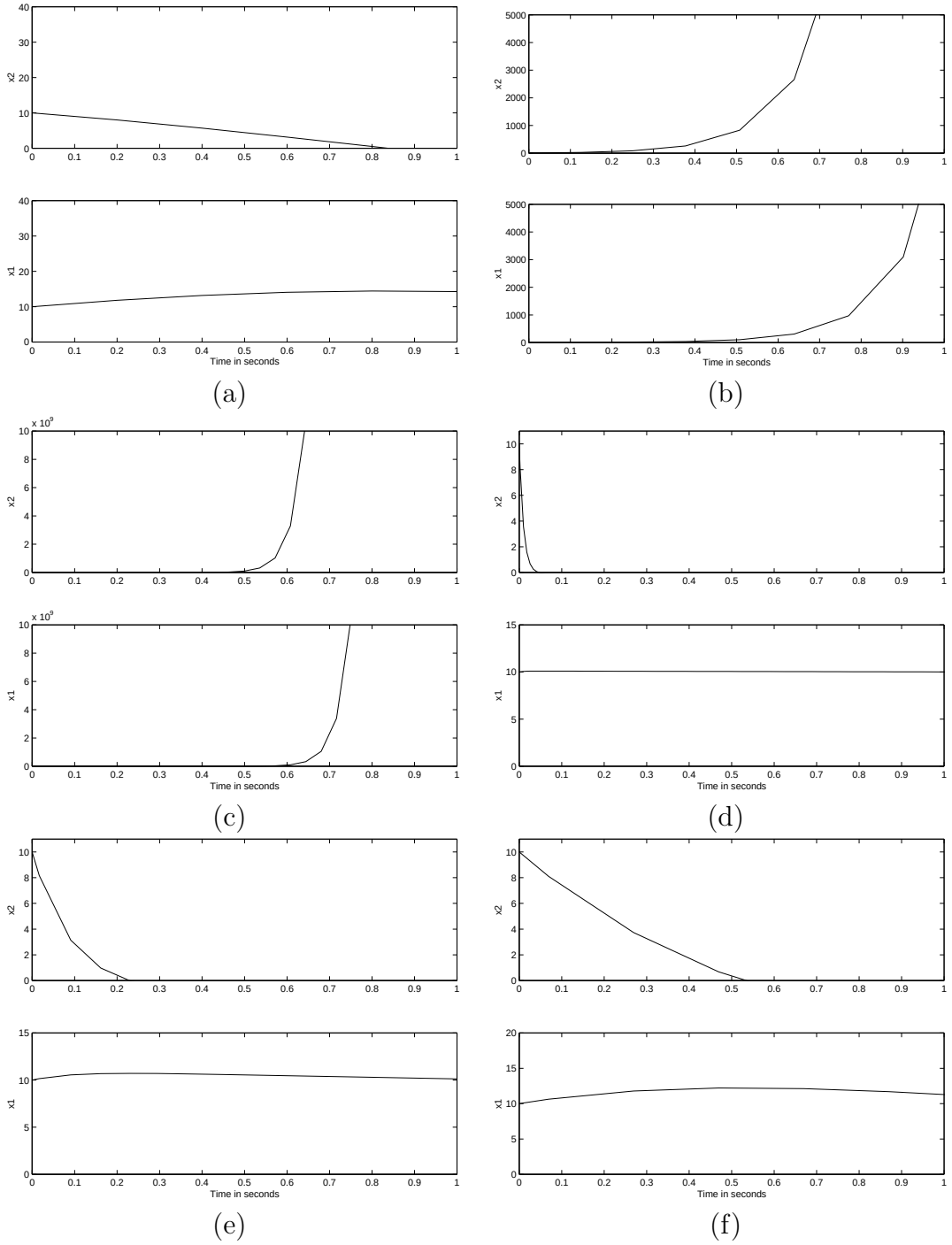


Figure 6: *Time responses of state variables. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*. With a sign function feedback and state variable feedbacks.*

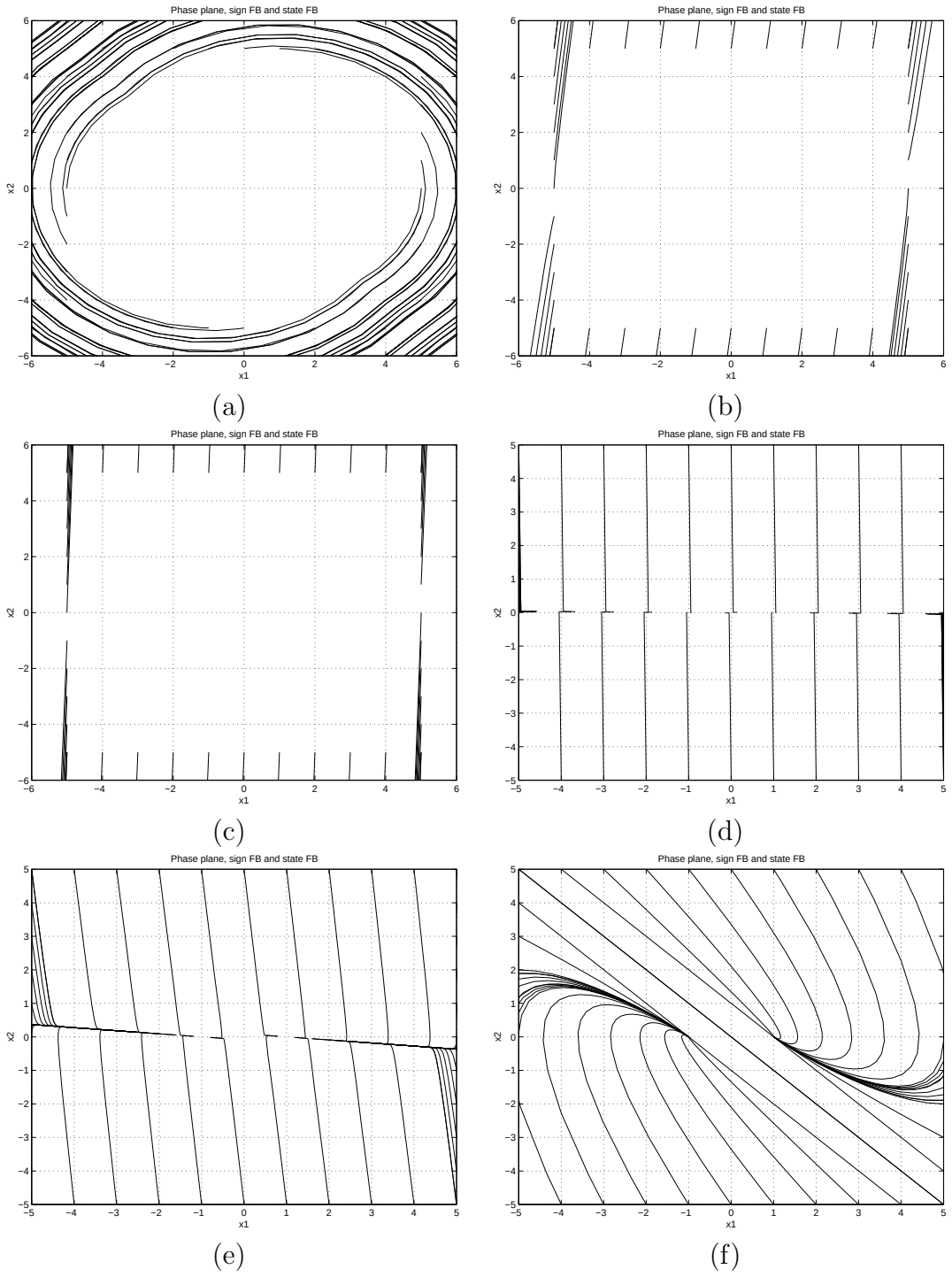


Figure 7: *State planes. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon =$ *and (f)* $\varepsilon = 1$*. With a sign function feedback and state variable feedbacks.*

Variable structure control

Let the hyperplane be described by

$$s = cx_1 + x_2, \tag{30}$$

which has the derivative

$$\dot{s} = c\dot{x}_1 + \dot{x}_2. \tag{31}$$

And let the input be

$$u = -K\text{sgn}\, s = -K\text{sgn}\,(cx_1 + x_2)\,, \quad K \in \mathcal{R}^+ \tag{32}$$

Choose a Lyapunov function as

$$V = \frac{1}{2}s^2. \tag{33}$$

Recall that the state equations are

$$\dot{x}_1 = x_2 \tag{34}$$

$$\dot{x}_2 = -\frac{1}{\varepsilon}x_2 - \frac{K}{\varepsilon}\text{sgn}\,(cx_1 + x_2)\,. \tag{35}$$

The derivative of the Lyapunov is then

$$\dot{V} = s\dot{s} \tag{36}$$

$$= (cx_1 + x_2)\left(cx_2 - \frac{1}{\varepsilon}x_2 - \frac{K}{\varepsilon}\text{sgn}\,(cx_1 + x_2)\right) \tag{37}$$

$$= c^2x_1x_2 - \frac{cx_1x_2}{\varepsilon} + cx_2^2 - \frac{x_2^2}{\varepsilon} - \frac{cKx_1}{\varepsilon}\text{sgn}\,(cx_1 + x_2) - \frac{Kx_2}{\varepsilon}\text{sgn}\,(cx_1 + x_2) \tag{38}$$

To observe the effect of ε on the stability of the controlled system, let

$$c = 2.7, \quad \text{and} \quad K = 3.3,$$

and plot $\dot{V}$ against x_1 and x_2 as shown in Figure 8. From Figure 8 whether $\dot{V} > 0$ or $\dot{V} < 0$ depends on the value of ε. For example, $\exists x$ such that $\dot{V} > 0$ when ε is 1.7 and 0.08, but $\dot{V} < 0\ \forall x$ when $\varepsilon = 0.3$.

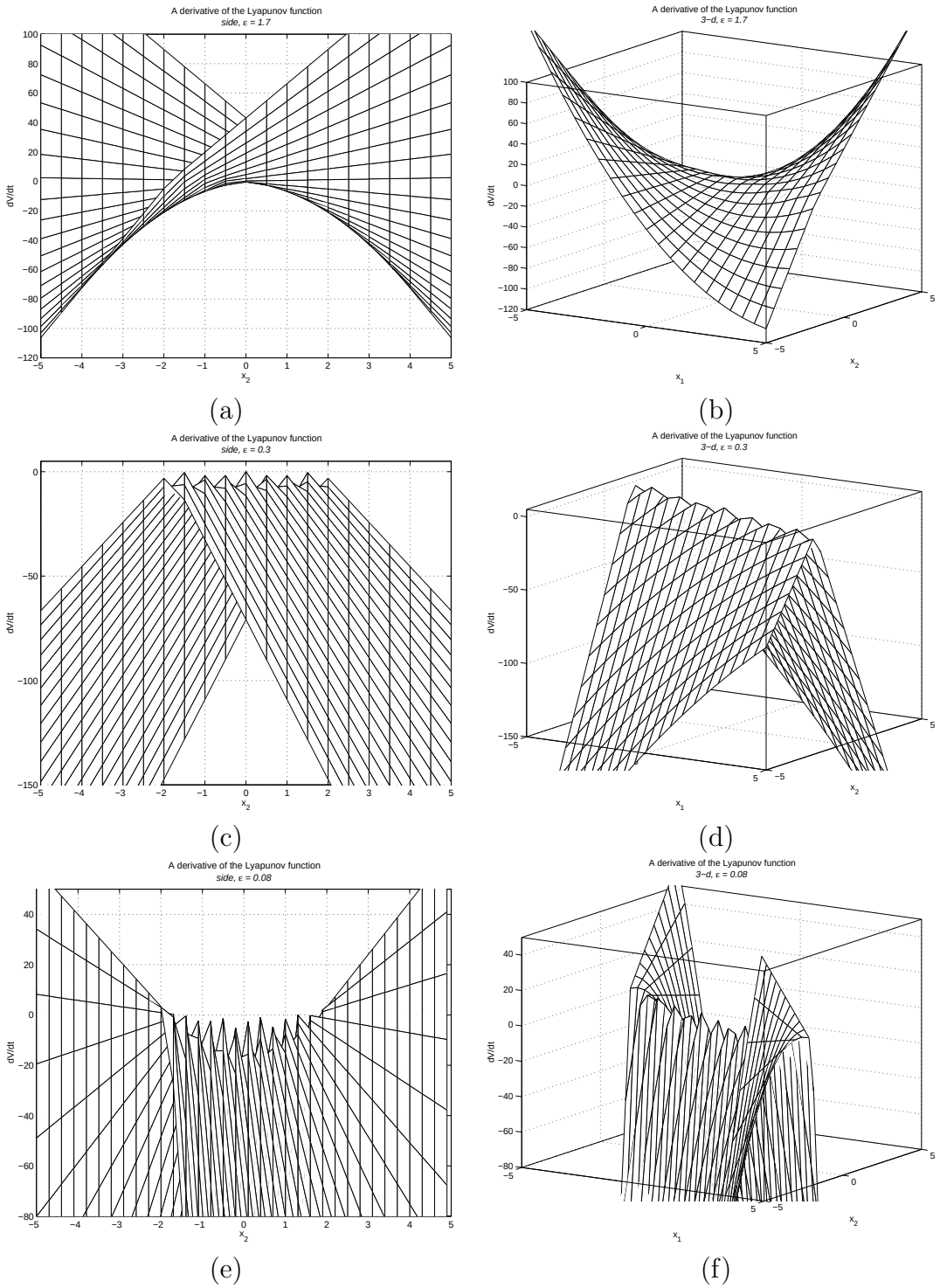


Figure 8: *Derivative of Lyapunov,* $\varepsilon = 1.7$ *for (a) and (b),* $\varepsilon = 0.3$ *for (c) and (d), and* $\varepsilon = 0.08$ *for (e) and (f)*

Figure 9 (a) and (b) shows the plots of the case of Figure 8 (c)-(d) and (e)-(f) respectively. Initial conditions were the Initial Conditions Set 1. Similar to Figure 9 (b), Figure 10 shows the case where $\varepsilon = 0.08$ but with the Initial Conditions Set 2 as the initial conditions instead. Despite the plot of $\dot{V}$ which shows that the system is unstable for $\varepsilon = 0.08$ the state planes of Figure 9 (b) and Figure 10 are stable.

Initial Condition Set 2 *The set of n points described by*

```
n = 0, where n is the number of initial conditions
i = 8
for j = -8, -5, -2, 1, 4, 7
     x1(0) = i, x2(0) = j, n = n+1
     x1(0) = -i, x2(0) = j, n = n+1
     x1(0) = j, x2(0) = i, n = n+1
     x1(0) = j, x2(0) = -i, n = n+1
endfor
```

The original system (Equation 2 and 3) has eigenvalues at $\lambda_1 = 0$ and $\lambda_2 = -12.5$, with the eigenvectors at

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -0.0797 \\ 0.9968 \end{bmatrix}$$

for λ_1 and λ_2 respectively.
The system is linear since

$$A \not\propto x, \tag{39}$$

ie A is a constant matrix. Also A is not a stability matrix since the condition

$$\mathrm{Re}\lambda_i < 0$$

is not satisfied. The equilibrium point at the origin is not unique as it has already been pointed out earlier that every point on the x_1-axis is an equilibrium point. The x_1-axis is a nontrivial null space of the matrix A
Change of variables

$$z = M^{-1}x, \quad M = [v_1, v_2] = \begin{bmatrix} 1 & -0.0797 \\ 0 & 0.9968 \end{bmatrix} \tag{40}$$

leads to

$$\begin{aligned} M\dot{z} &= AMz \\ \dot{z} &= M^{-1}AMz \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -12.5 \end{bmatrix} z, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & -12.5 \end{bmatrix}. \end{aligned}$$

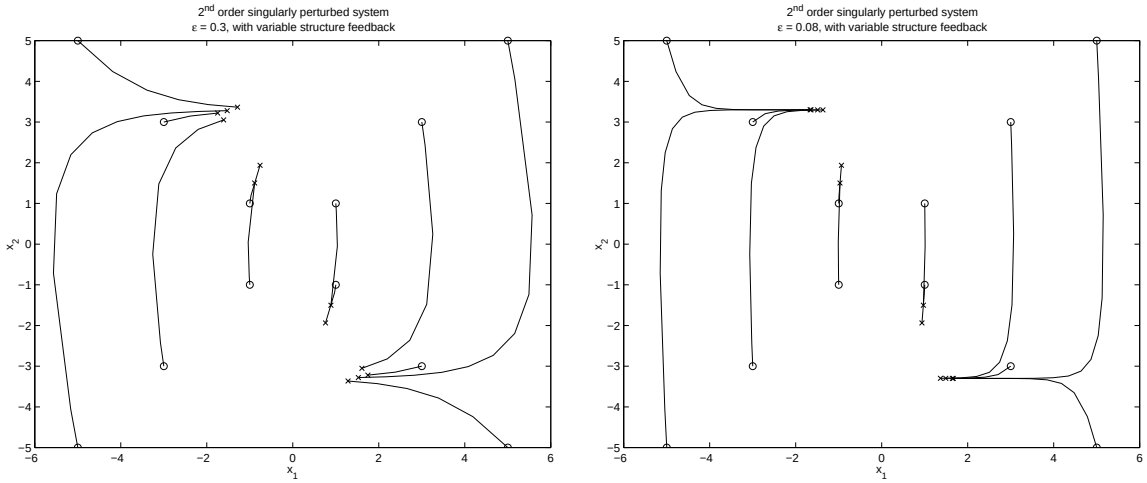
The solution to this system is

$$\dot{z}_1(t) = z_1(0), \quad z_2(t) = z_2(0)e^{-12.5t} = z_2(0)e^{\lambda_2 t}.$$

The fact that A is not a stability matrix can be seen by trying to solve the Lyapunov equation

$$PA + A^T P = -Q, \tag{41}$$

where Q is a positive definite symmetric matrix. The Lyapunov equation yields no solution for this system. ♠



(a) (b)

Figure 9: *State planes (a)* $\varepsilon = 0.3$*, (b)* $\varepsilon = 0.08$

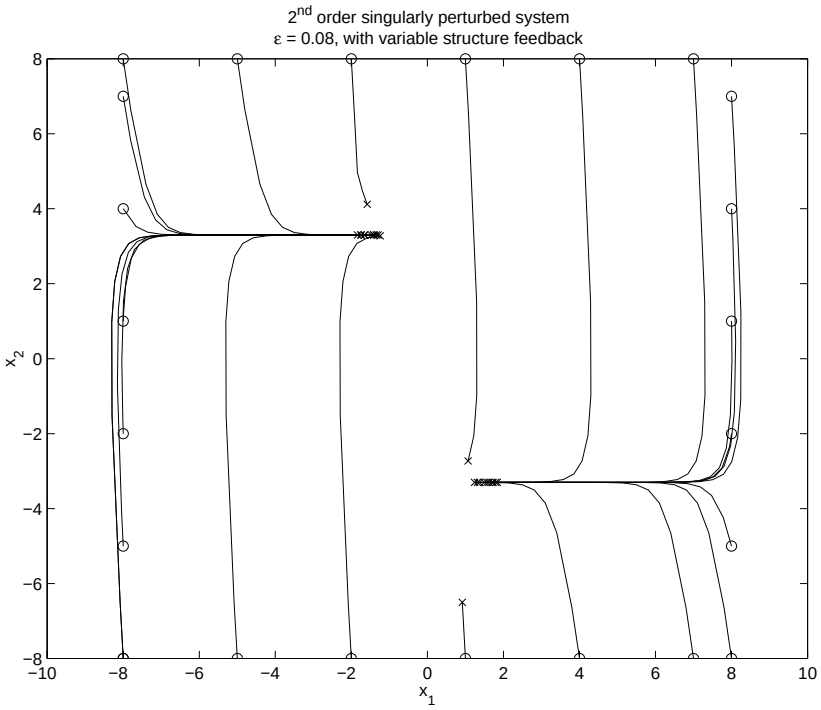


Figure 10: *State planes* $\varepsilon = 0.08$

Figure 11 (a) to (f) shows some more of the plot of $\dot{V}$ with various parameters. Figure 11 (a) and (b) have

$$c = 2.7, \quad \text{and} \quad K = 3.3$$

as in Figure 8 but here

$$\varepsilon < 0.$$

Figure 11 (c) to (f) have

$$K = 100$$

while $\varepsilon = 1.7$ and $\varepsilon = 0.08$ for (c)-(d) and (e)-(f) respectively. Figure 11 shows that when $\varepsilon < 0$ the controlled system is unstable (Figure 11 (a)-(b)). For $\varepsilon > 0$ a larger value of K gives a wider range of ε which the controlled system is stable. In other words the larger the value of gain K of the control input is, the more robust to ε this control scheme will be. Compare Figure 8 (a), (b), (e), and (f) with Figure 11 (c), (d), (e), and (f) respectively.

N.B. One way to stabilize the origin of the system is perhaps to solve the equation

$$A^T PE + E^T PA - \left(E^T PB + S\right) R^{-1} \left(B^T PE + S^T\right) + Q = 0, \tag{42}$$

or equivalently

$$F^T PE + E^T PF - E^T PBR^{-1}B^T PE + Q - SR^{-1}S^T = 0, \tag{43}$$

where $F = A - BR^{-1}S^T$, Q and R are symmetric positive definite matrice, and the system is written in the form

$$E\dot{x} = Ax + Bu.$$

As an example, let

$$Q = R = E = I, \quad S = 0$$

will lead to

$$P = \begin{bmatrix} 12.62 & 1 \\ 1 & 0.1194 \end{bmatrix}.$$

The feedback using this method is then

$$u = -Gx, \quad G = R^{-1}\left(B^T PE + S^T\right). \tag{44}$$

The gain matrix for this system would be

$$G = [\ 1 \quad 0.1194\ ].$$

The closed-loop eigenvalues could then be computed from

$$(A - BG)\, V = EV\Gamma,$$

where

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where λ_i is the i^{th} eigenvalue and v_i its corresponding eigenvector. In solving the Riccati equation the Frobenius norm of the relative residual matrix is 3.14×10^{-16}. A simulation result of the stabilized system is shown in Figure 12. The initial condition is the Initial Condition Set 1. ♣

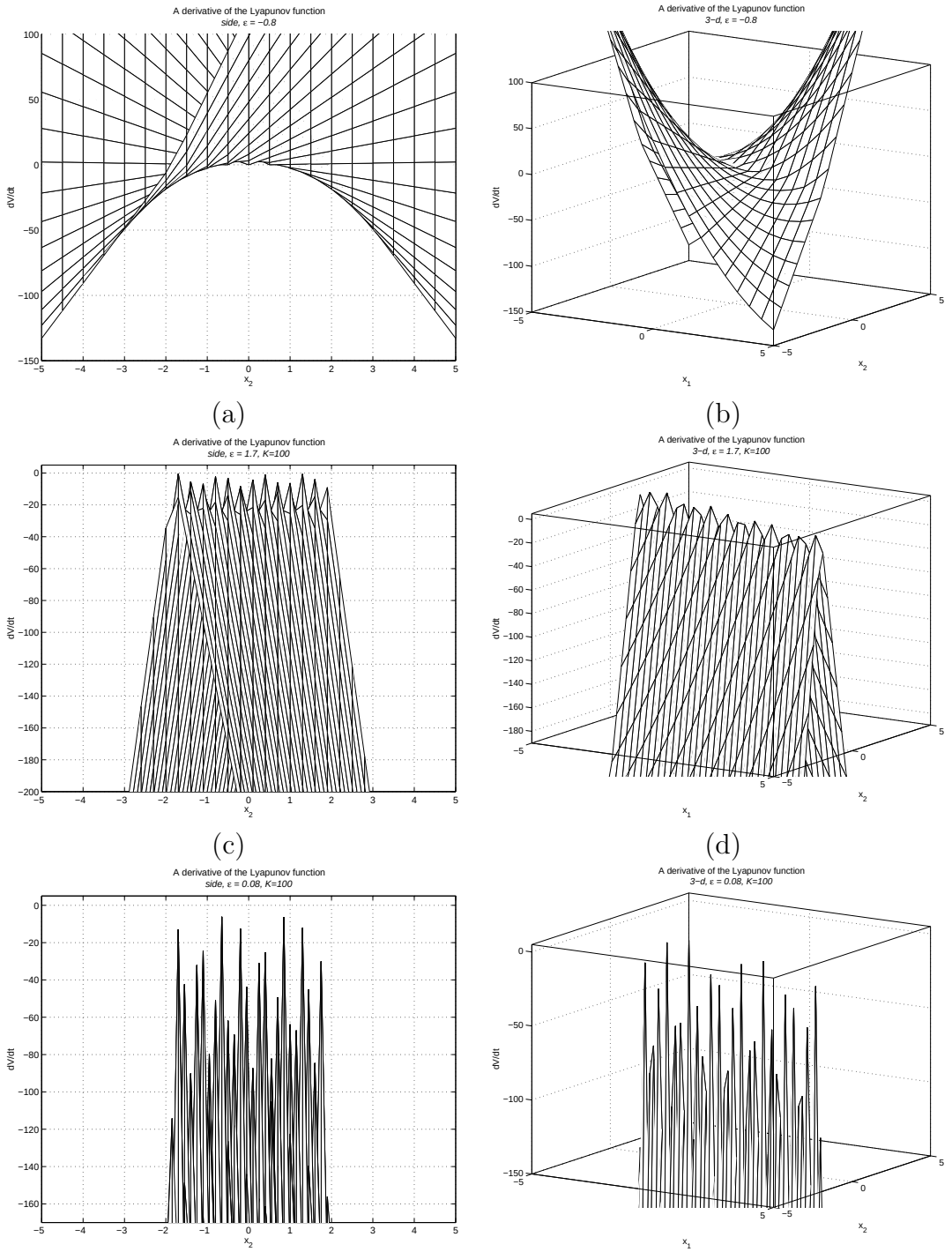


Figure 11: *Derivative of Lyapunov,* $\varepsilon = -0.8$ *for (a) and (b),* $\{\varepsilon = 1.7, K = 100\}$ *for (c) and (d), and* $\{\varepsilon = 0.08, K = 100\}$ *for (e) and (f)*

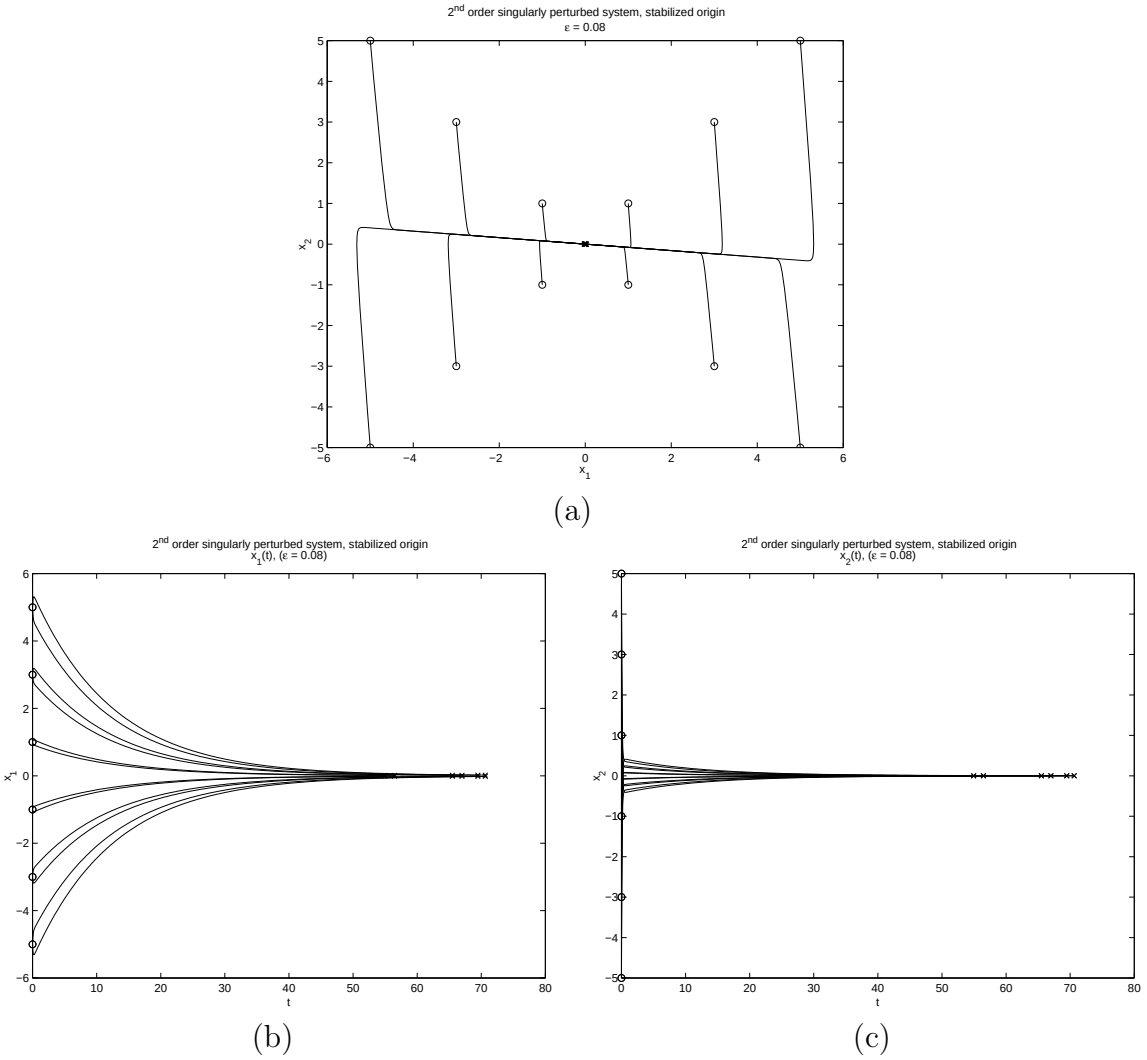
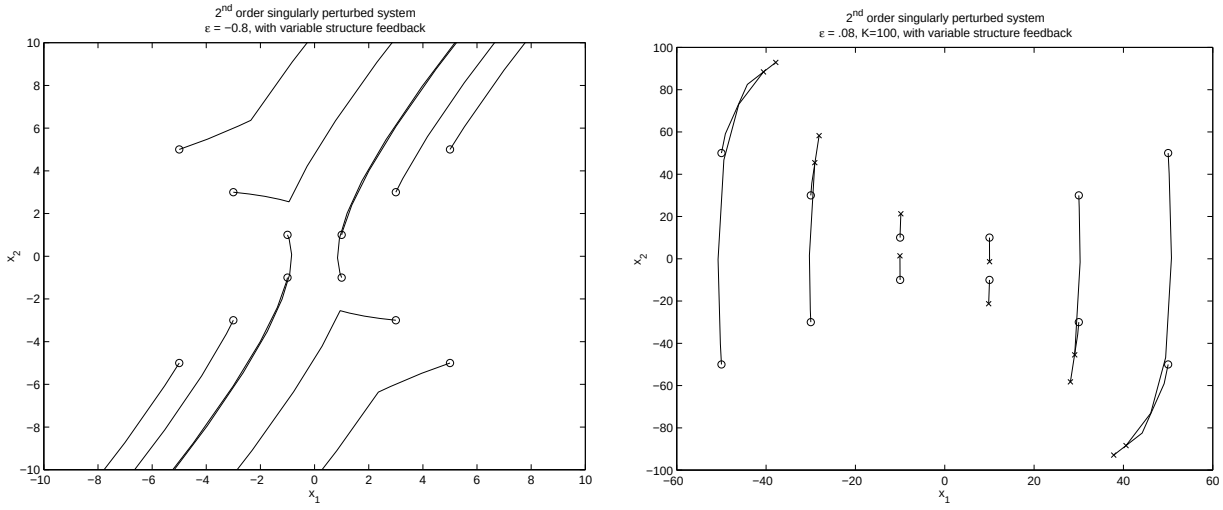


Figure 12: *System with stabilized origin (a) phase plane, (b) $x_1(t)$, and (c) $x_2(t)$ $\varepsilon = 0.08$*

Figure 13 (a) and (b) are the plots of the cases of Figure 11 (a)-(b) and (e)-(f) respectively. Initial conditions were the Initial Conditions Set 1 for 13 (a), and Initial Conditions Set 3 for Figure 13 (b). When $\varepsilon < 0$ the system is unstable. Figure 14 (a) and (b) are when $K < 0$ in $u = -K\text{sgn}s$. From Figure 14 (a) there is a domain of attraction outside of which the trajectory does not converge. From simulations with different values of parameters it can be seen that the phase planes when $\varepsilon > 0$ while $K > 0$ in $u = -K\text{sgn}s$ have the same characteristic which is represented by Figure 14 (b). The higher value of the feedback gain K only changes the value of x_2 which the trajectory converges onto, but the phase plane characteristic is still the same.

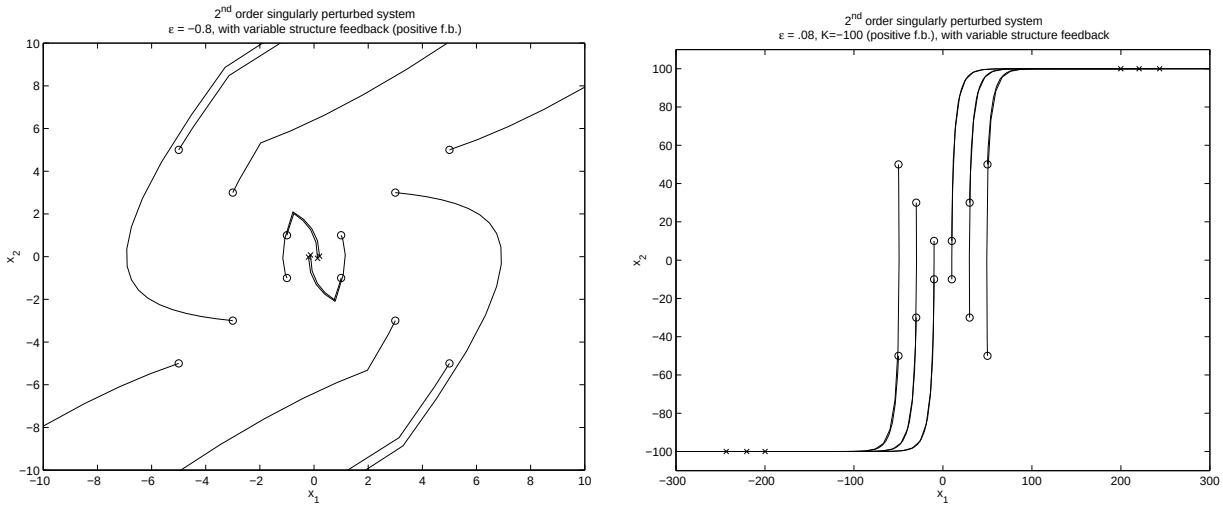
Initial Condition Set 3 *The set of n points described by*

```
n = 0
for i = -50, -30,..., 50
     x1(0) = i, x2(0) = i
     x1(0) = i, x2(0) = -i
     n = n+2
endfor
```



(a) (b)

Figure 13: *State planes (a)* $\varepsilon = 0.3$*, (b)* $\varepsilon = 0.08$


2nd order singularly perturbed system
ε = −0.8, with variable structure feedback (positive f.b.)
x1
x2
2nd order singularly perturbed system
ε = .08, K=−100 (positive f.b.), with variable structure feedback
x1
x2

(a) (b)

Figure 14: *State planes (a)* $\varepsilon = -0.8$*, (b)* $\varepsilon = 0.08 \quad K = -100$*, ie positive variable structure feedback*

Another second order singularly perturbed system

Consider the problem

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0 \tag{45}$$

which can be represented as state equations

$$\dot{x}_1 = x_2 \tag{46}$$
$$\varepsilon \dot{x}_2 = -x_1 - x_2 + u. \tag{47}$$

Some knowledge about the system

The only equilibrium point is the origin. ($f(x) = 0 \Rightarrow x_1 = x_2 = 0$) The Jacobian matrix is

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=(x_1,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{bmatrix}.$$

The Eigenvalues are therefore

$$\frac{1}{2\varepsilon}\left(-1 \pm \sqrt{1-4\varepsilon}\right).$$

The Eigenvalues are shown in Figure 15 graphically as real parts and absolute imaginary part. And

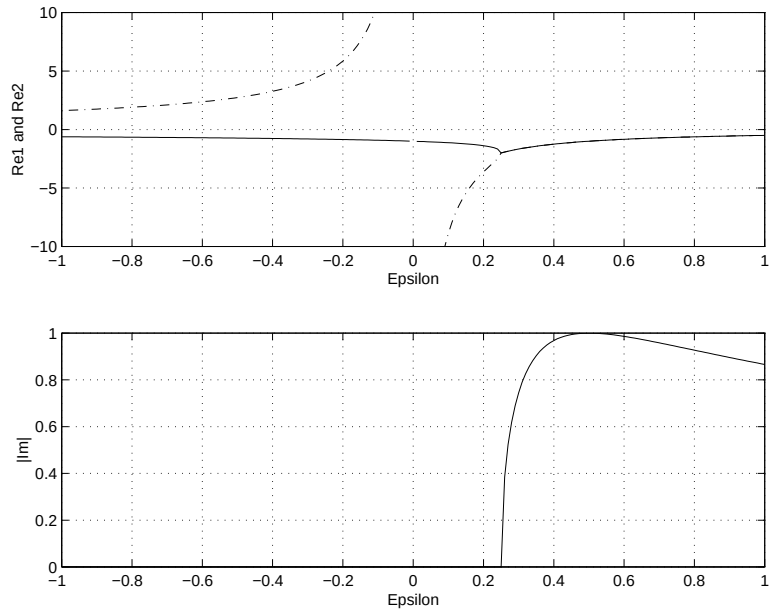


Figure 15: *Real and imaginary parts of the eigenvalues*

the nature of the trajectories can be outlined in Table 2.
Table 3 shows A, P, λ_1, λ_2, v_1, and v_2 for various ε.

Values of ε	Behaviour of trajectory
$-\infty < \varepsilon < 0$	saddle point
$\varepsilon = 0$	system degenerated
$0 < \varepsilon \leq 0.25$	stable node
$\varepsilon > 0.25$	stable focus

Table 2: *Qualitative behaviour of the trajectories.*

$\varepsilon < 0$

$(\cdot)$	$\varepsilon = -1$	$\varepsilon = -0.1$	$\varepsilon = -0.02$
A	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 10 & 10 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 50 & 50 \end{bmatrix}$
P	$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.455 & -0.5 \\ -0.5 & 0.45 \end{bmatrix}$	$\begin{bmatrix} 0.4902 & -0.5 \\ -0.5 & 0.49 \end{bmatrix}$
λ_1	-0.618	-0.9161	-0.9808
λ_2	1.618	10.9161	50.9808
v_1	$\begin{bmatrix} 0.8507 \\ -0.5257 \end{bmatrix}$	$\begin{bmatrix} -0.7374 \\ 0.6755 \end{bmatrix}$	$\begin{bmatrix} -0.7139 \\ 0.7002 \end{bmatrix}$
v_2	$\begin{bmatrix} 0.5257 \\ 0.8507 \end{bmatrix}$	$\begin{bmatrix} -0.0912 \\ -0.9958 \end{bmatrix}$	$\begin{bmatrix} -0.0196 \\ -0.9998 \end{bmatrix}$

$\varepsilon > 0$

$(\cdot)$	$\varepsilon = 0.01$	$\varepsilon = 0.1$	$\varepsilon = 1$
A	$\begin{bmatrix} 0 & 1 \\ -100 & -100 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -10 & -10 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$
P	$\begin{bmatrix} 0.505 & -0.5 \\ -0.5 & 0.505 \end{bmatrix}$	$\begin{bmatrix} 0.555 & -0.5 \\ -0.5 & 0.55 \end{bmatrix}$	$\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$
λ_1	-1.0102	-1.127	$-0.5 + 0.866\,i$
λ_2	-98.9898	-8.873	$-0.5 - 0.866\,i$
v_1	$\begin{bmatrix} 0.7035 \\ -0.7107 \end{bmatrix}$	$\begin{bmatrix} 0.6637 \\ -0.748 \end{bmatrix}$	$\begin{bmatrix} 0.6124 - 0.3536\,i \\ 0.7071\,i \end{bmatrix}$
v_2	$\begin{bmatrix} -0.0101 \\ 0.9999 \end{bmatrix}$	$\begin{bmatrix} -0.112 \\ 0.9937 \end{bmatrix}$	$\begin{bmatrix} 0.6124 + 0.3536\,i \\ -0.7071\,i \end{bmatrix}$

Table 3: *Solution of the Lyapunov equation, the eigenvalues and the eigenvectors*

Response with various values of ε

When $u = 0$ the time response of state variables and the state planes are shown in Figure 16 and Figure 17 respectively. The result shows that the system is stable when $\varepsilon > 0$ and is unstable otherwise. According to Table 2, Figure 16 (a)-(c) have an equilibriumpoint as a saddle point, (d)-(e) are stable nodes, and (f) is a stable focus.

A sensitivity analysis of the system was carried out as follows. [Kha96] Letting $\varepsilon = 1$ be the nominal value of ε turns Equation 46 and 47 ($u = 0$) into

$$\dot{x}_1 = x_2 \tag{48}$$
$$\dot{x}_2 = -x_1 - x_2. \tag{49}$$

And the system with ε incorporated can be rewritten as

$$\dot{x}_1 = x_2 \tag{50}$$
$$\dot{x}_2 = -\frac{1}{\varepsilon}x_1 - \frac{1}{\varepsilon}x_2. \tag{51}$$

The Jacobian matrix $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \lambda}$ are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{bmatrix} \tag{52}$$

and

$$\frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial \varepsilon} = \begin{bmatrix} 0 \\ \frac{x_1}{\varepsilon^2} + \frac{x_2}{\varepsilon^2} \end{bmatrix} \tag{53}$$

respectively. These Jacobian matrice evaluated at the nominal ε yield

$$\left.\frac{\partial f}{\partial x}\right|_{nominal} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{54}$$

and

$$\left.\frac{\partial f}{\partial \lambda}\right|_{nominal} = \begin{bmatrix} 0 \\ x_1 + x_2 \end{bmatrix}. \tag{55}$$

The augmented equation is

$$\begin{array}{rclcrcl} \dot{x} & = & f(t,x,\lambda), & & x(t_0) & = & x_0 \\ \dot{x}_\lambda & = & \left[\frac{\partial f(t,x,\lambda)}{\partial x}\right] x_\lambda + \left[\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right], & & x_\lambda(t_0) & = & 0 \end{array} \quad .http://straitstimes.asia1.com.sg/lee/leeindex.html \tag{56}$$

Let

$$x_\lambda = x_\varepsilon = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial x_1}{\partial \varepsilon} & \frac{\partial x_2}{\partial \varepsilon} \end{bmatrix}^T\right|_{nominal}.$$

(analysis to be continued on page 29.)

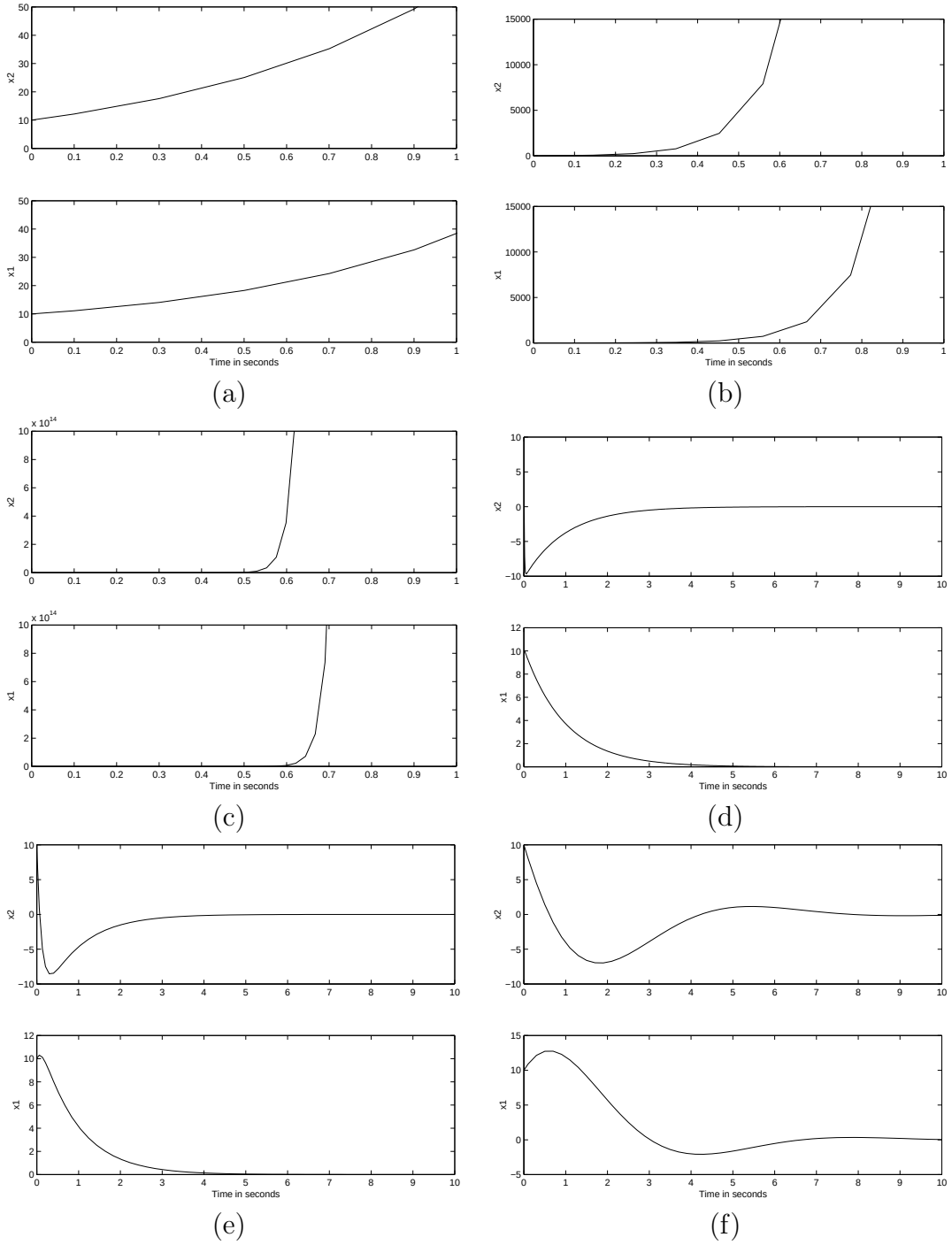


Figure 16: *State variables. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.02$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*.*

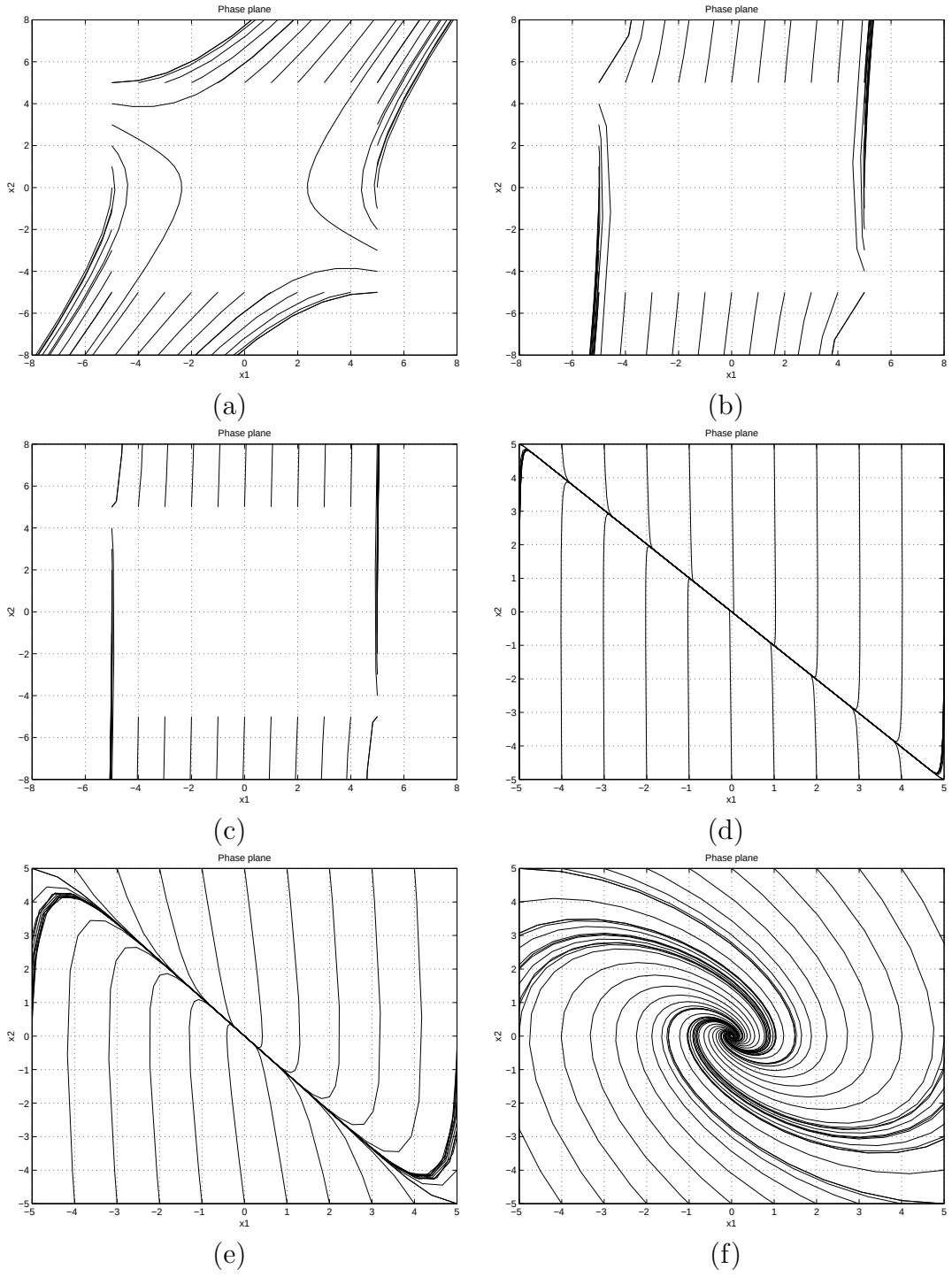


Figure 17: *State planes. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.02$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*.*

With a sign function in feedback

When

$$u = \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$$

the time response of state variables and the state planes are shown in Figure 19 and Figure 20 respectively.

(Continued from page 26, sensitivity analysis.)

the sensitivity estimation part of Equation 56 is verbosely described as

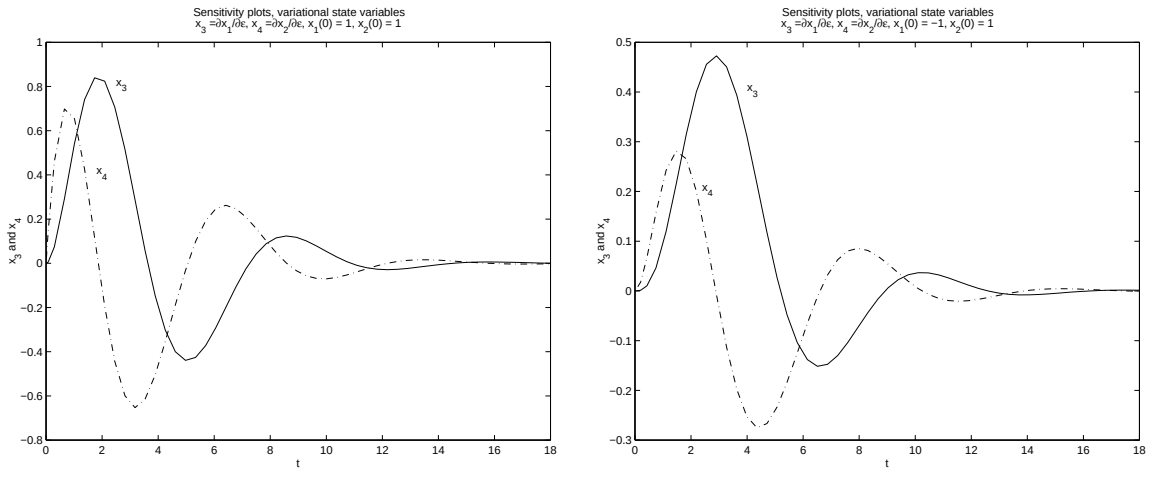
$$\dot{x}_3 = \left.\frac{\partial f_1}{\partial x_1}\right|_{nominal} x_3 + \left.\frac{\partial f_1}{\partial x_2}\right|_{nominal} x_4 + \left.\frac{\partial f_1}{\partial \varepsilon}\right|_{nominal} \tag{57}$$

$$\dot{x}_4 = \left.\frac{\partial f_2}{\partial x_1}\right|_{nominal} x_3 + \left.\frac{\partial f_2}{\partial x_2}\right|_{nominal} x_4 + \left.\frac{\partial f_2}{\partial \varepsilon}\right|_{nominal} \tag{58}$$

and Equation 56 becomes

$$\begin{array}{lll} \dot{x}_1 = x_2 & x_1(0) = x_{10} \\ \dot{x}_2 = -x_1 - x_2 & x_2(0) = x_{20} \\ \dot{x}_3 = x_4 & x_3(0) = 0 \\ \dot{x}_4 = -x_3 - x_4 + x_1 + x_2 & x_4(0) = 0 \end{array} \tag{59}$$

The result of simulating this system is show in Figure 18. Figure 18 (a) differs from Figure 18 (b) in the initial conditions given for x_1 and x_2. Both figures show that x_1 is slightly more sensitive to $\Delta\varepsilon$ than x_2. In other words, $\frac{\partial x_1}{\partial \varepsilon} > \frac{\partial x_2}{\partial \varepsilon}$.



(a) (b)

Figure 18: *Sensitivity plot of* $x_3 = \frac{\partial x_1}{\partial \varepsilon}$ *and* $x_3 = \frac{\partial x_2}{\partial \varepsilon}$*, (a)*$x_1(0) = x_2(0) = 1$*, (b)*$x_1(0) = -1, x_2(0) = 1$

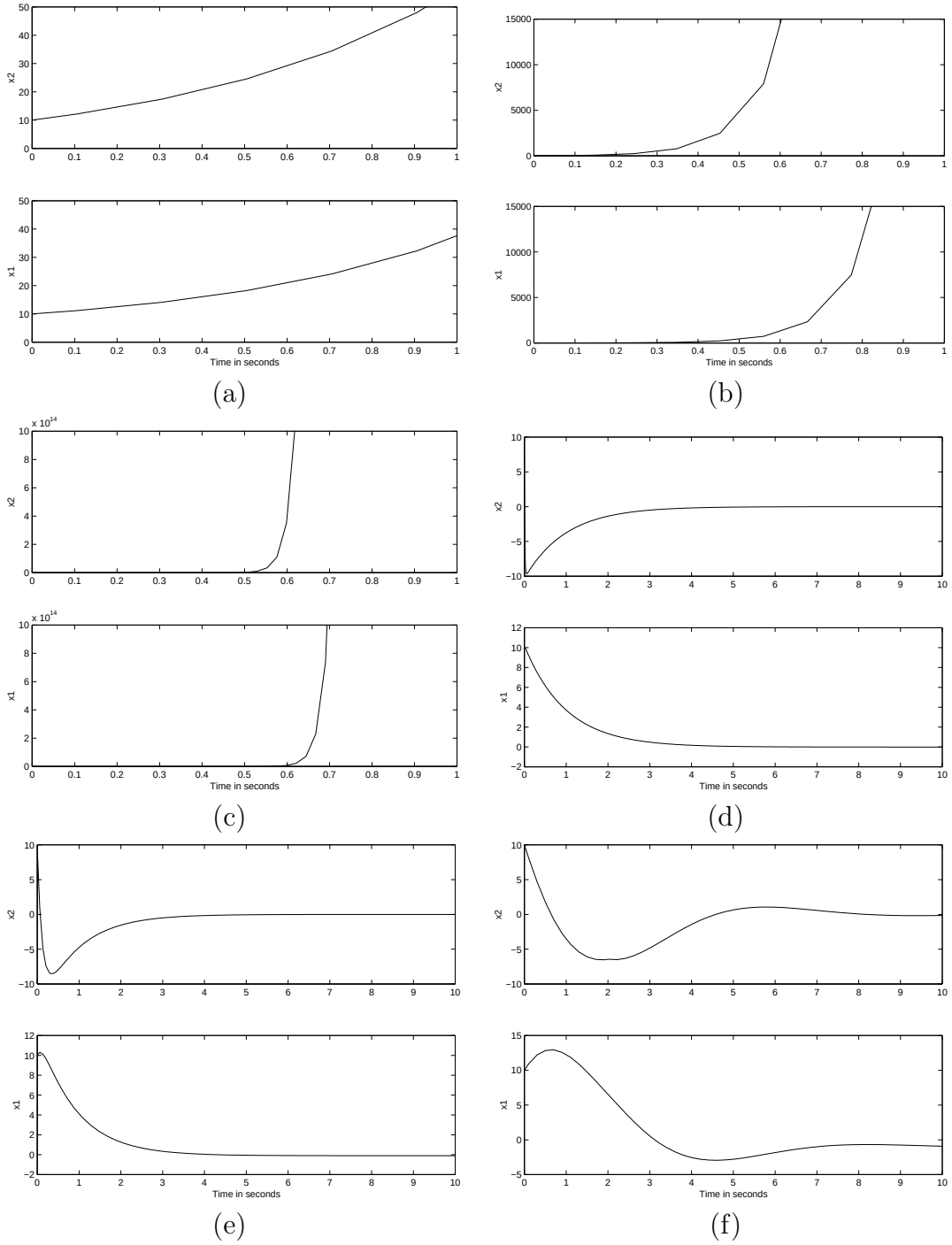


Figure 19: *State variables. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.02$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*. With a sign function feedback.*

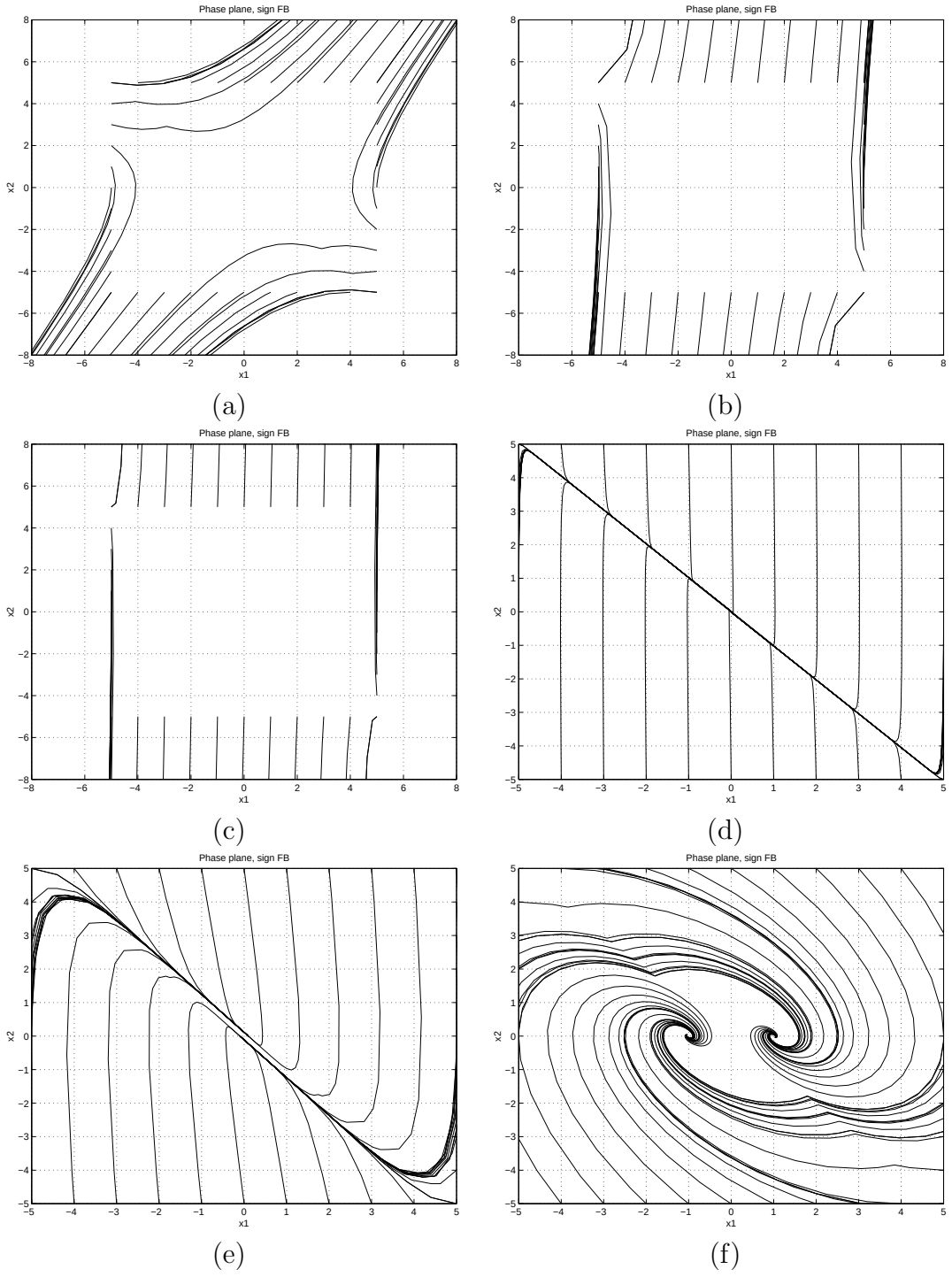


Figure 20: *State planes. (a)* $\varepsilon = -1$*, (b)* $\varepsilon = -0.1$*, (c)* $\varepsilon = -0.02$*, (d)* $\varepsilon = 0.01$*, (e)* $\varepsilon = 0.1$ *and (f)* $\varepsilon = 1$*. With a sign function feedback.*

ε	$\dot{V}$'s plots	phase plane plots	nature of the equilibrium point
-1	Figure 21	Figure 24 (a), (b)	saddle point
0.1	Figure 22	Figure 24 (c), (d)	stable node
0.7	Figure 23	Figure 24 (e), (f)	stable focus

Table 4: $\dot{V}$ *of various values of* ε

Let the input be

$$u = -K\text{sgn } s \tag{60}$$

where s is a hyperplane and

$$s = cx_1 + x_2, \quad c \in \mathcal{R}. \tag{61}$$

Then

$$\begin{aligned} \dot{s} &= c\dot{x}_1 + \dot{x}_2 \\ &= cx_2 - \frac{x_1}{\varepsilon} - \frac{x_2}{\varepsilon} - \frac{K\text{sgn}\,(cx_1 + x_2)}{\varepsilon}. \end{aligned} \tag{62}$$

Choose the Lyapunov function as

$$V = \frac{1}{2}s^2. \tag{63}$$

Then

$$\begin{aligned} \dot{V} &= s\dot{s} \\ &= (cx_1 + x_2)\left(cx_2 - \frac{x_1}{\varepsilon} - \frac{x_2}{\varepsilon} - \frac{K\text{sgn}\,(cx_1 + x_2)}{\varepsilon}\right) \end{aligned} \tag{64}$$

Let

$$c = 0.8, \quad \text{and} \quad K = 15.2,$$

the derivative of the Lyapunov function (Equation 64) is plotted the way described by Table 4.

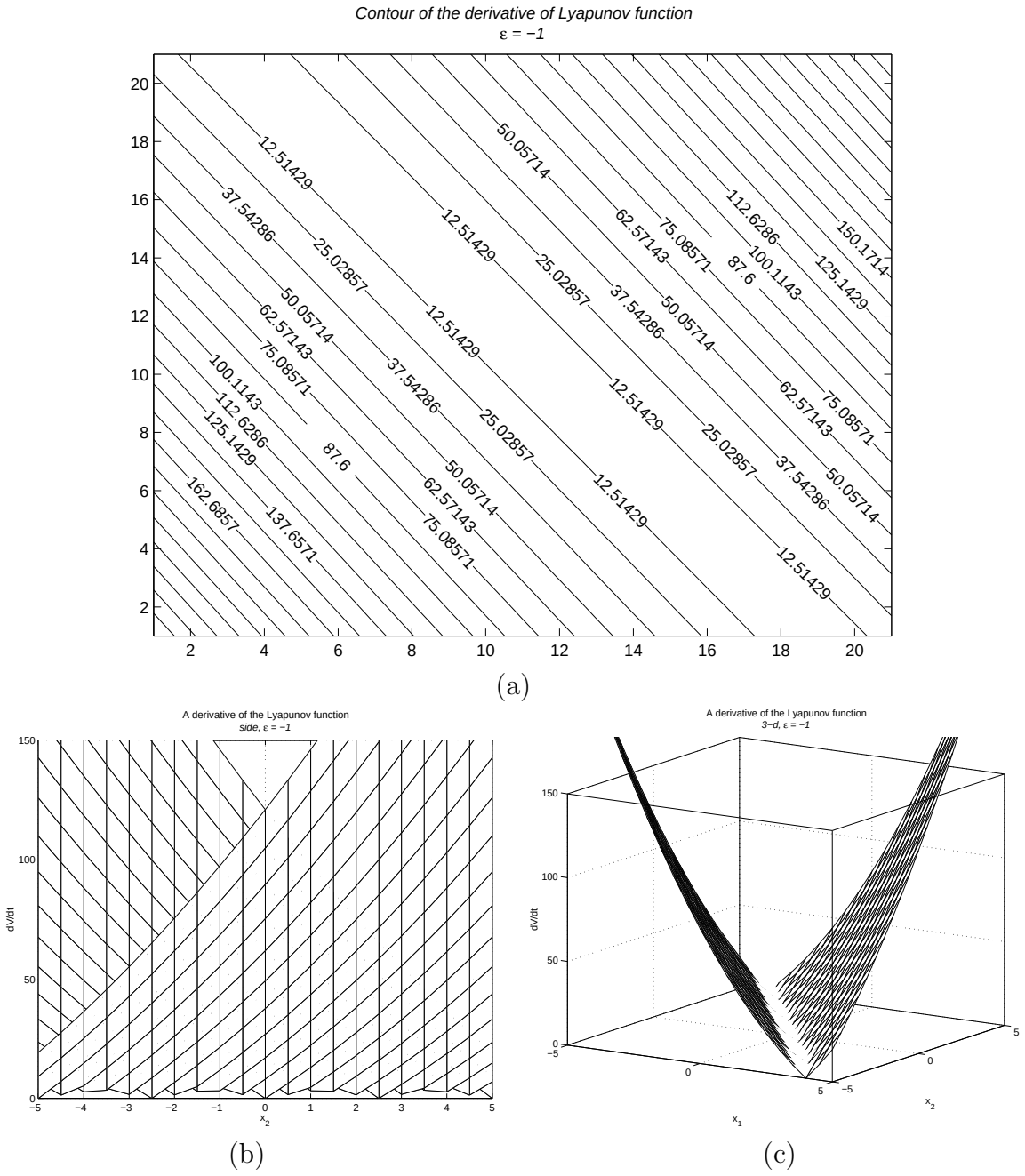


Figure 21: $\dot{V}$ *(a) contour plot, (b) and (c) phase plots* $\varepsilon = -1$

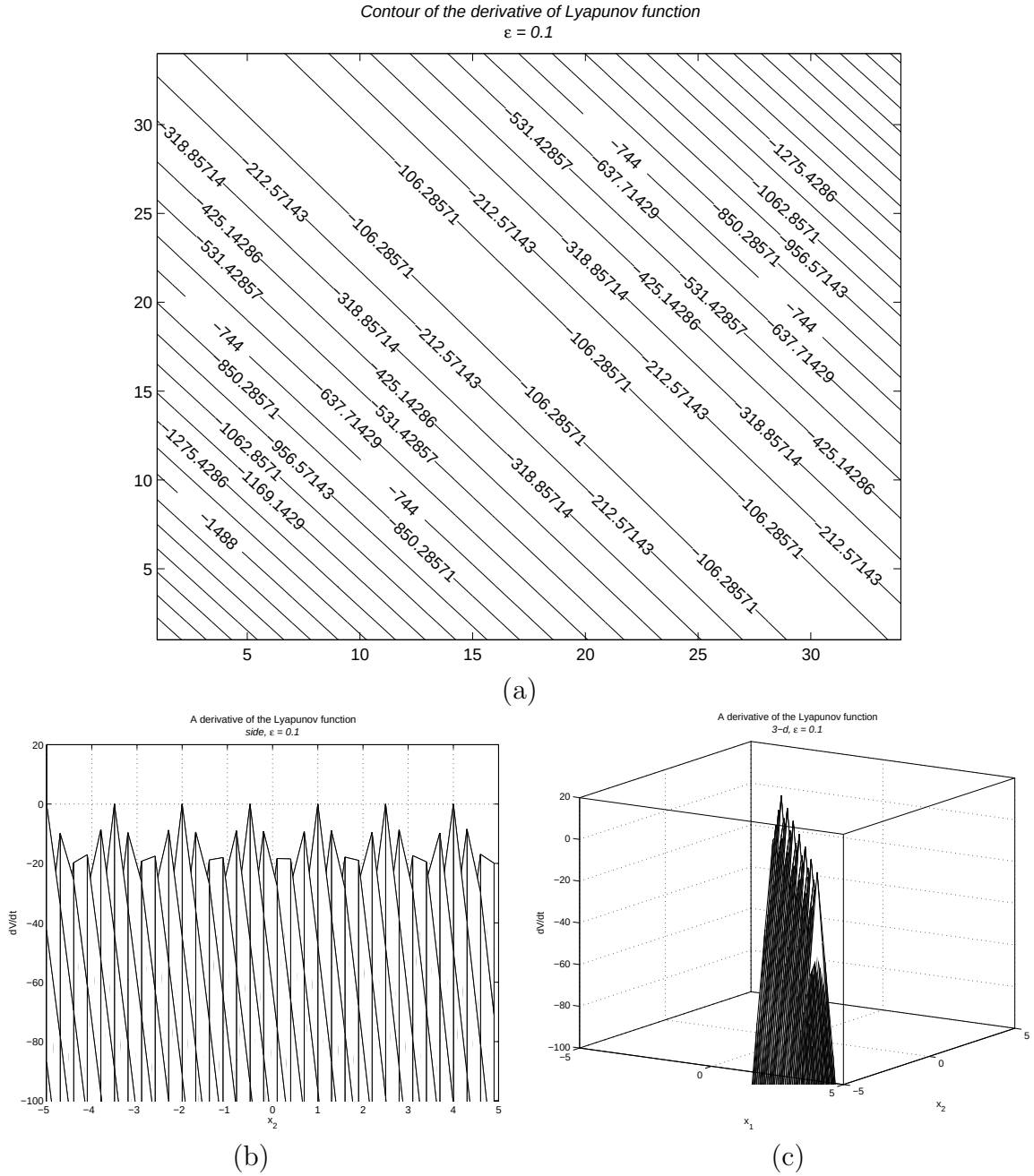


Figure 22: $\dot{V}$ *(a) contour plot, (b) and (c) phase plots* $\varepsilon = 0.1$

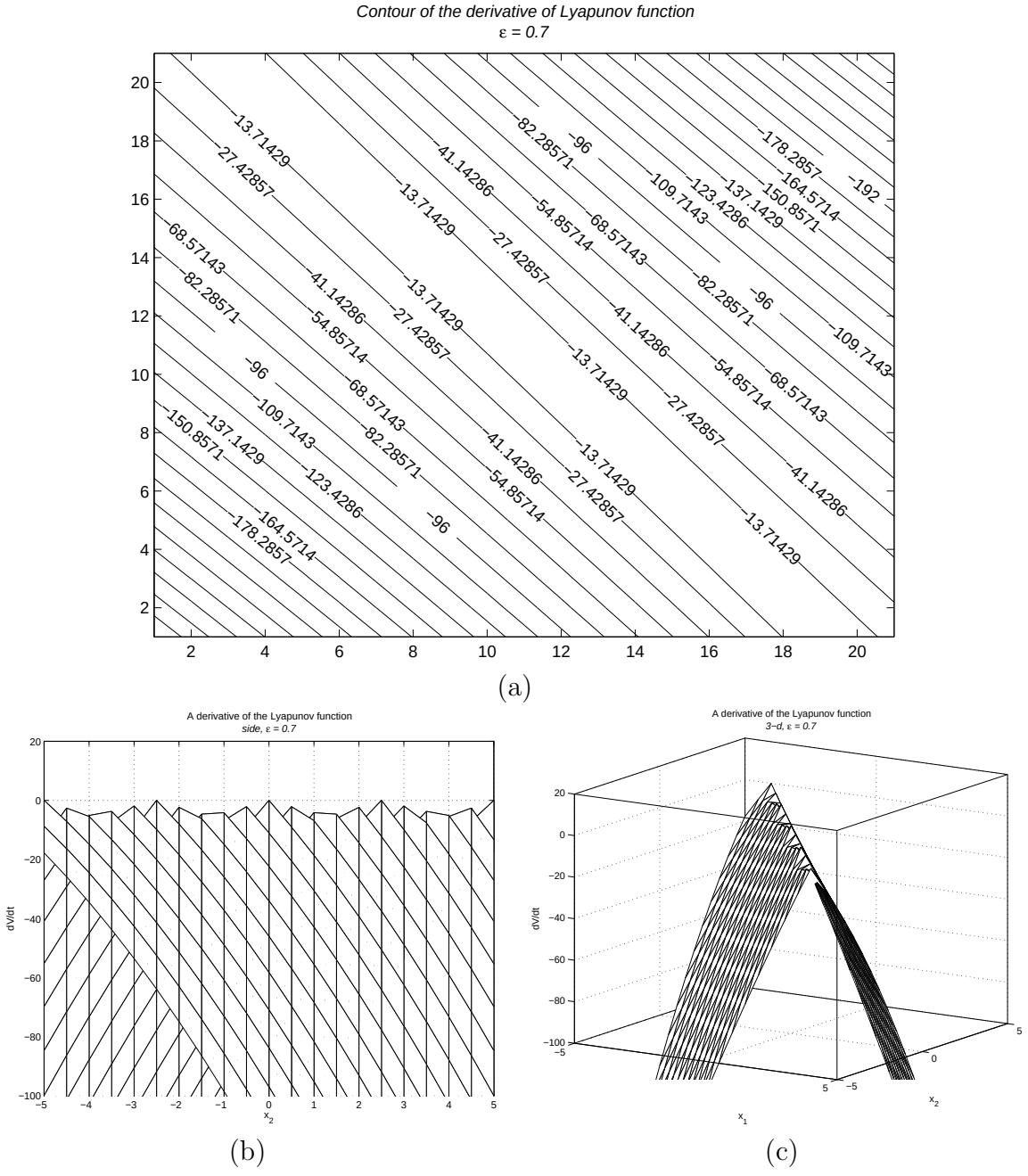


Figure 23: $\dot{V}$ *(a) contour plot, (b) and (c) phase plots* $\varepsilon = 0.7$

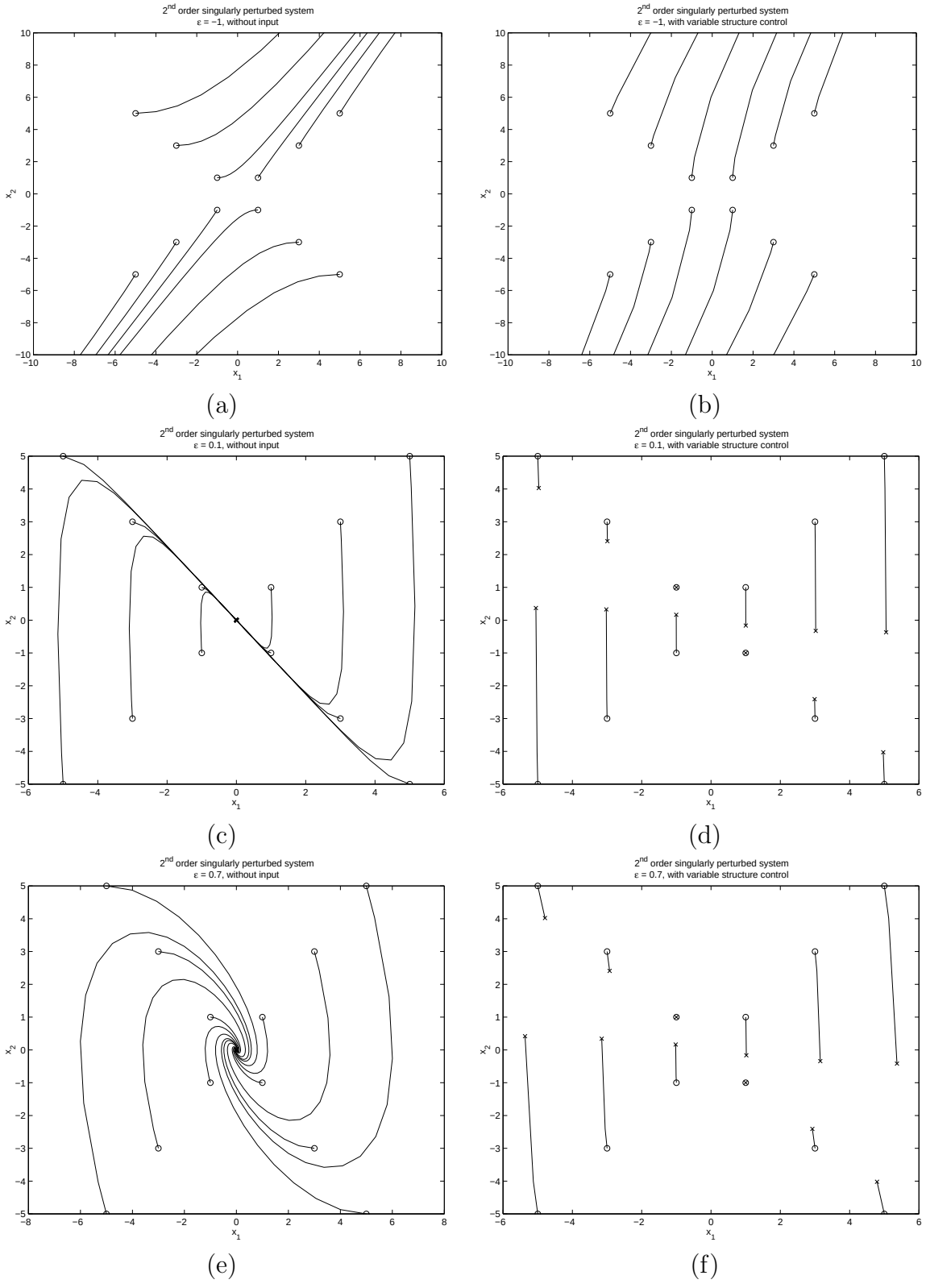


Figure 24: *Phase plane of the* 2*-nd order system (a), (b)* $\varepsilon = -1$*; (c), (d)* $\varepsilon = 0.1$*; (e), (f)* $\varepsilon = 0.7$

Fourth order singularly perturbed system

Consider next the singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}z + B_1u, \tag{65}$$
$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_2u. \tag{66}$$

Let $A_{11} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Some knowledge about the system

As this is a linear system the eigenvalues and eigenvectors can be obtained from the matrix A. For $\varepsilon = 1$,

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

The four eigenvalues are

$$\lambda_1 = 3.7321, \quad \lambda_2 = 1.5 + 0.866\,i, \quad \lambda_3 = 1.5 - 0.866\,i, \quad \lambda_4 = 0.2679,$$

and the eigenvectors are

$$v_1 = \begin{bmatrix} 0.445 \\ -0.2225 \\ 0.6894 \\ 0.5265 \end{bmatrix}, v_2 = \begin{bmatrix} 0.3449 + 0.0776\,i \\ -0.1343 + 0.5975\,i \\ 0.1343 - 0.5975\,i \\ -0.1053 - 0.3375\,i \end{bmatrix}, v_3 = \begin{bmatrix} 0.3449 - 0.0776\,i \\ -0.1343 - 0.5975\,i \\ 0.1343 + 0.5975\,i \\ -0.1053 + 0.3375\,i \end{bmatrix}, v_4 = \begin{bmatrix} -0.6388 \\ 0.3194 \\ 0.6701 \\ -0.2025 \end{bmatrix}.$$

The state equations are converted into a transfer function and the root locus plotted in Figure 25.

$$TF = \frac{s^4 - 4s^3 + 7s^2 - 6s + 3.521 \times 10^{-16}}{s^4 - 7s^3 + 16s^2 - 15s + 3}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, C = I_{4\times4}, D = 1$$

The matrix A for a general case where $\varepsilon \neq 1$ is

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ \frac{2}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} \end{bmatrix}.$$

The movement of the four eigenvalues is portrayed in Figure 26. The values of ε used for plotting this figure are

$$\varepsilon = -2, -1.91, -1.82, -1.73, \ldots, \varepsilon_{i-1} + 0.09, \ldots, 1.87, 1.96.$$

In the light of Figure 26 the system is always unstable no matter what value ε may take since there are always eigenvalues with a positive real-part. With very small value of ε ($|\varepsilon| \ll 1$) the change in system characteristic is abrupt. When $\varepsilon = 0$ the system degenerates.

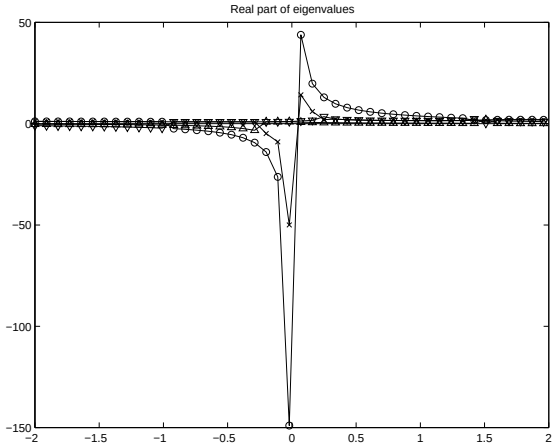


Figure 25: *Root locus of the 4^{th}-order system when ε*

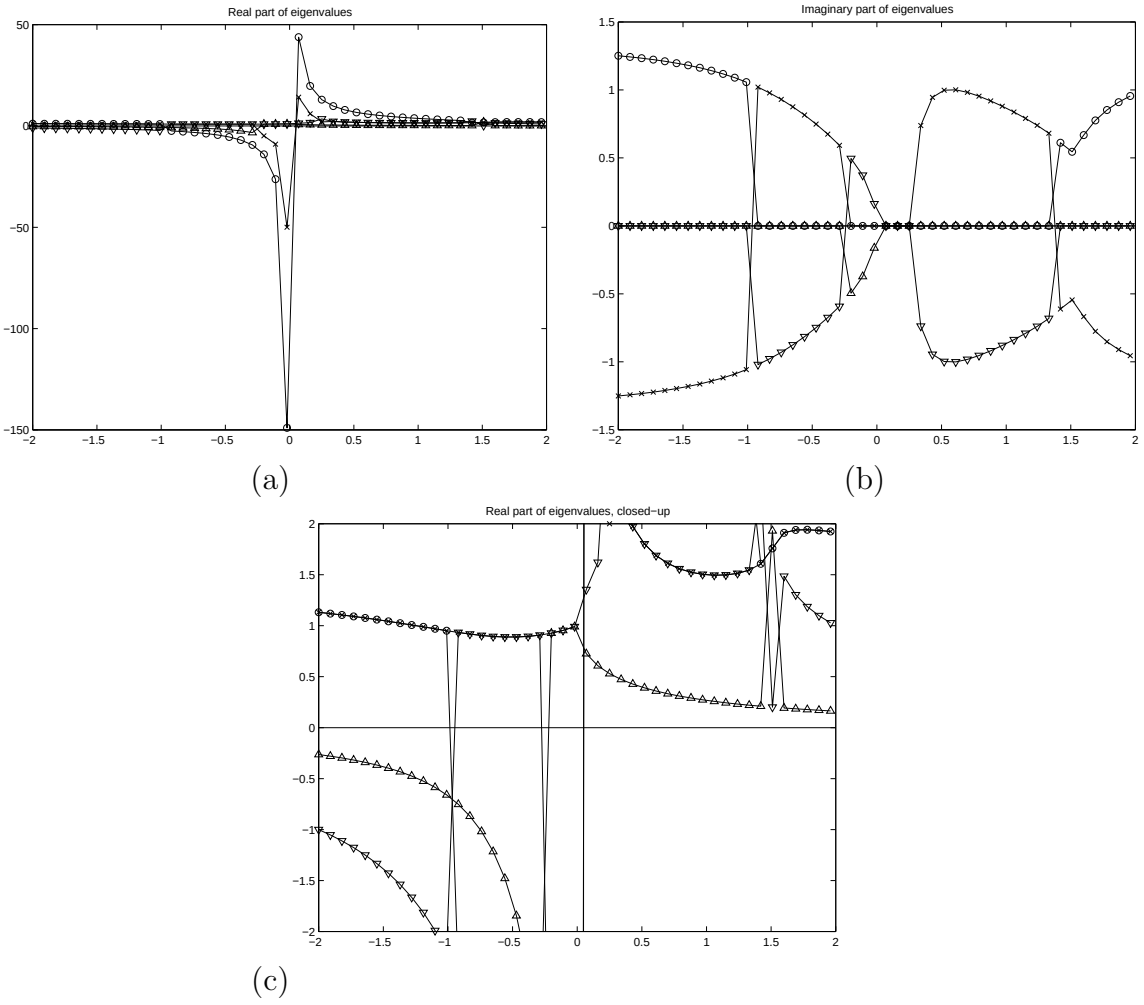


Figure 26: *Movement of eigenvalues with ε (a) real part, (b) imaginary part, (c) real part (closed-up)*

Response with various values of ε

When $\varepsilon = -1$ the state variables and the state plane are shown in Figure 27 and Figure 28 respectively. Here x_3 and x_4 are z_1 and z_2 respectively.

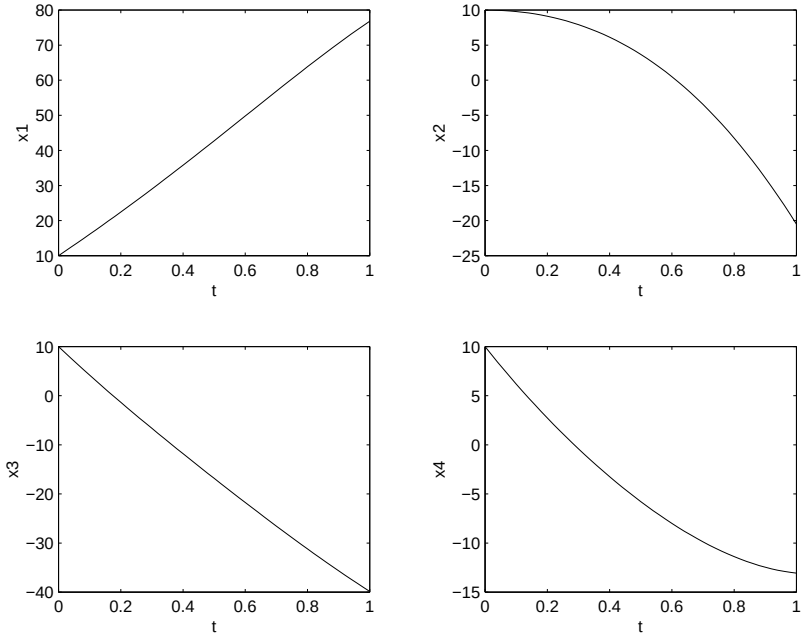


Figure 27: *State variable,* $\varepsilon = -1$.

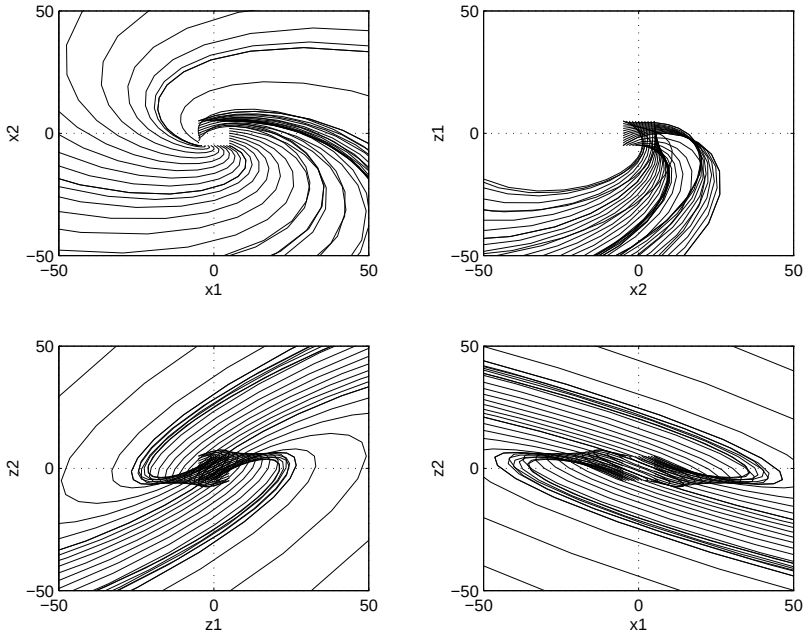


Figure 28: *State plane,* $\varepsilon = -1$.

When $\varepsilon = -0.1$ the state variables and the state plane are shown in Figure 29 and Figure 30 respectively.

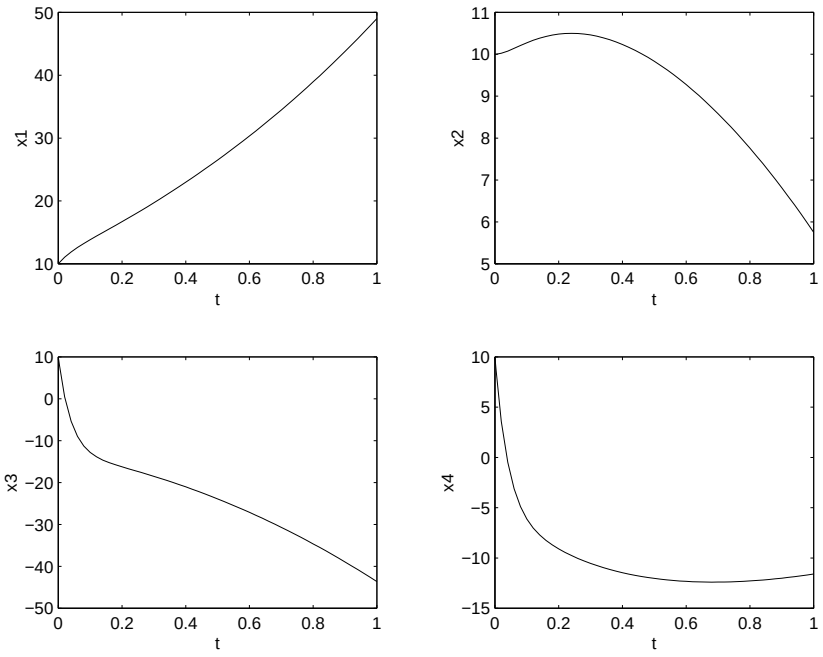


Figure 29: *State variable,* $\varepsilon = -0.1$.

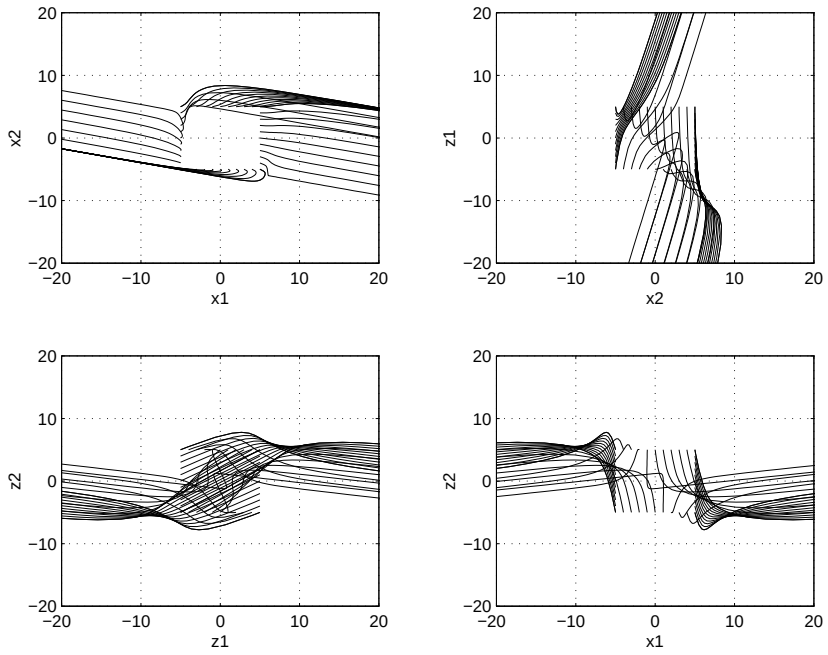


Figure 30: *State plane,* $\varepsilon = -0.1$.

When $\varepsilon = -0.01$ the state variables and the state plane are shown in Figure 31 and Figure 32 respectively.

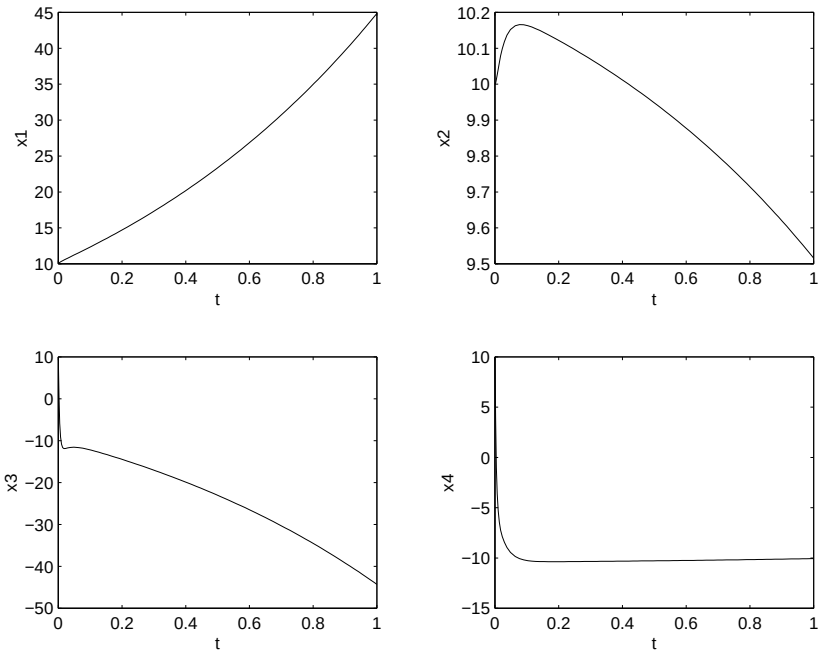


Figure 31: *State variable,* $\varepsilon = -0.01$.

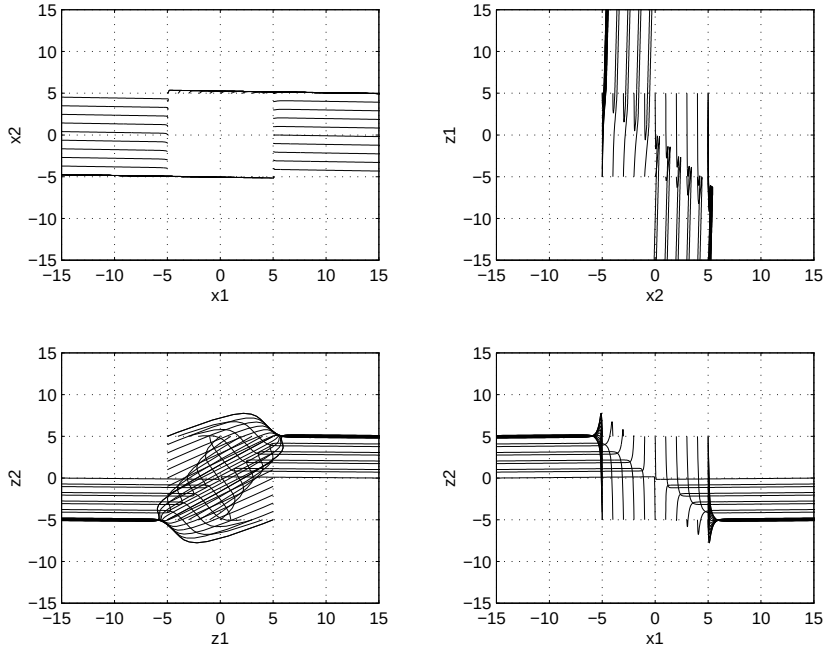


Figure 32: *State plane,* $\varepsilon = -0.01$.

When $\varepsilon = -0.001$ the state variables and the state plane are shown in Figure 33 and Figure 34 respectively.

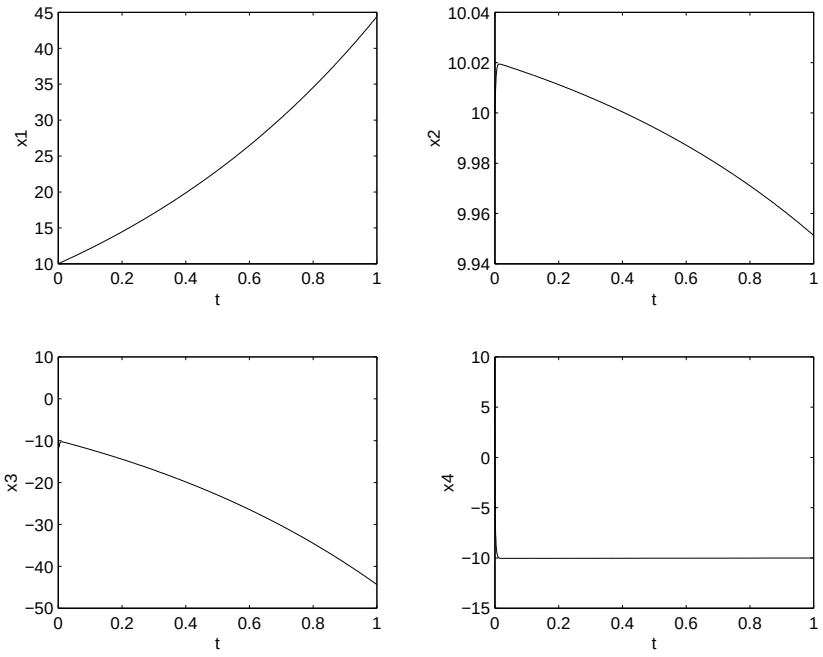


Figure 33: *State variable,* $\varepsilon = -0.001$.

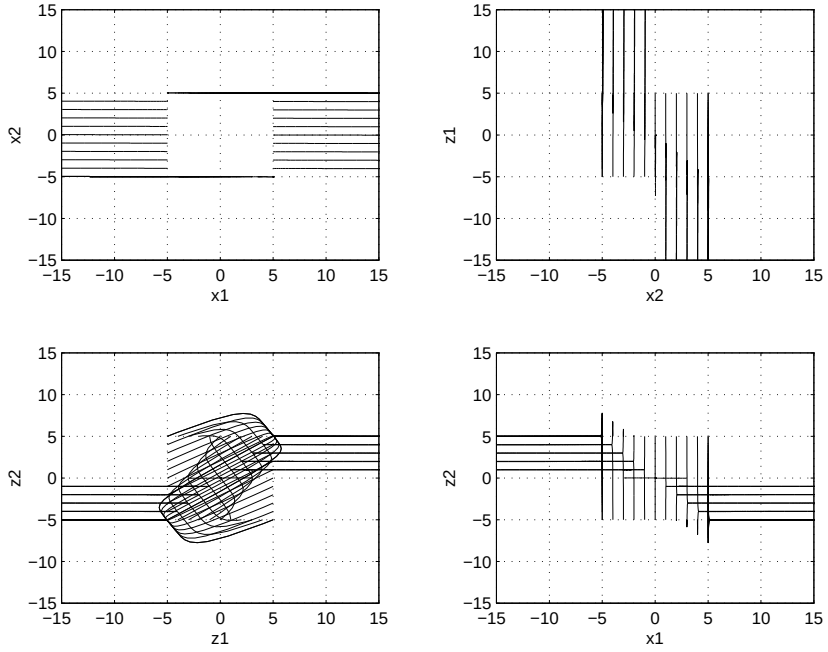


Figure 34: *State plane,* $\varepsilon = -0.001$.

When $\varepsilon = 0.01$ the state variables and the state plane are shown in Figure 35 and Figure 36 respectively.

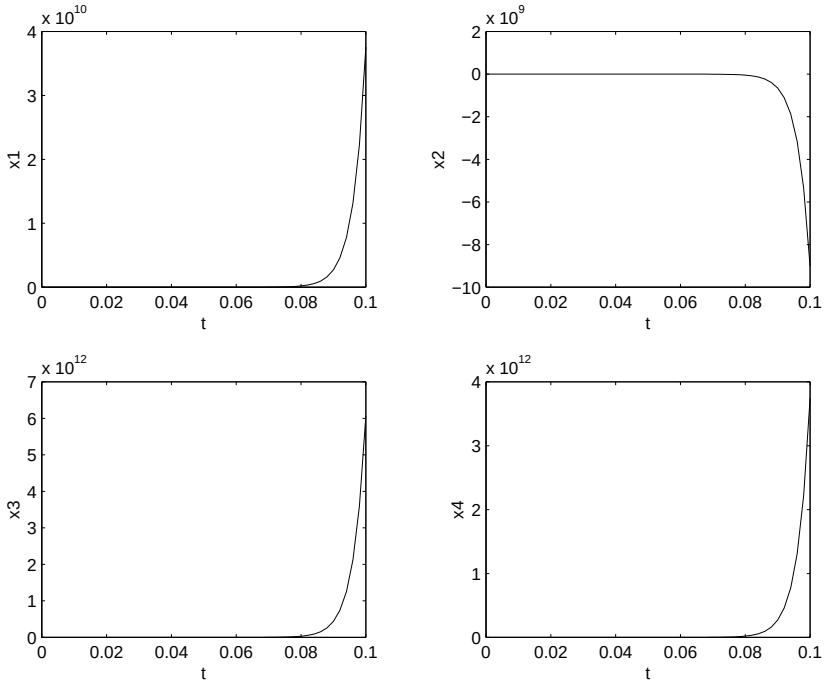


Figure 35: *State variable,* $\varepsilon = 0.01$.

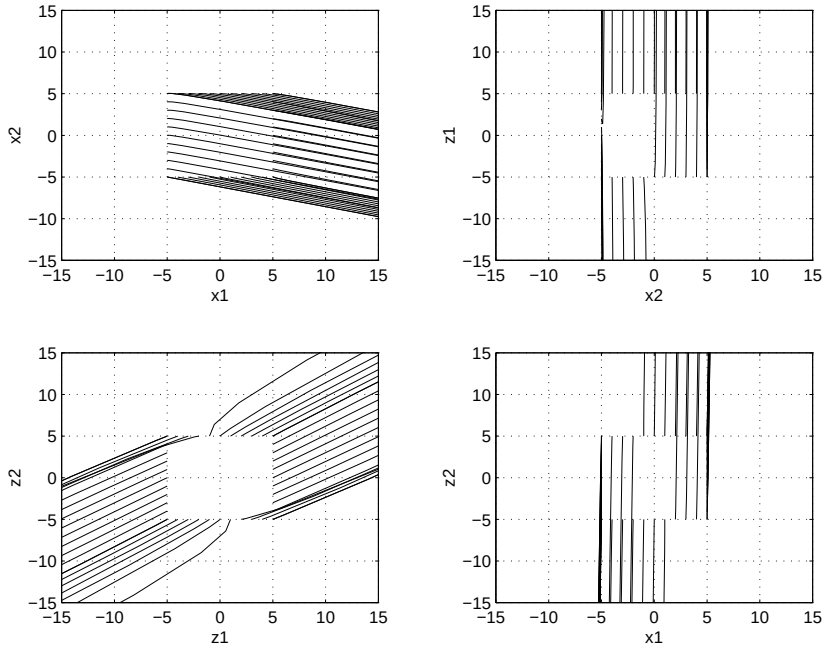


Figure 36: *State plane,* $\varepsilon = 0.01$.

When $\varepsilon = 0.1$ the state variables and the state plane are shown in Figure 37 and Figure 38 respectively.

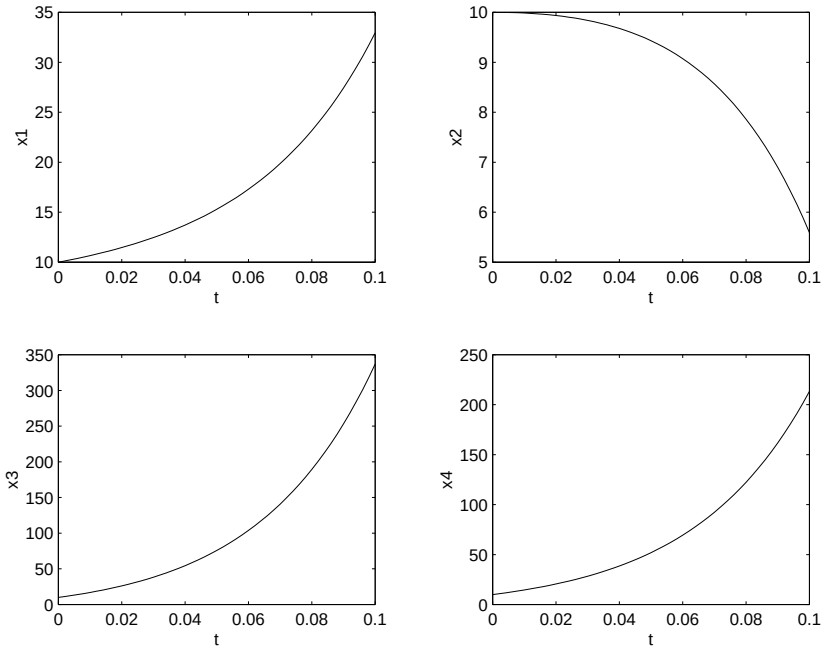


Figure 37: *State variable,* $\varepsilon = 0.1$.

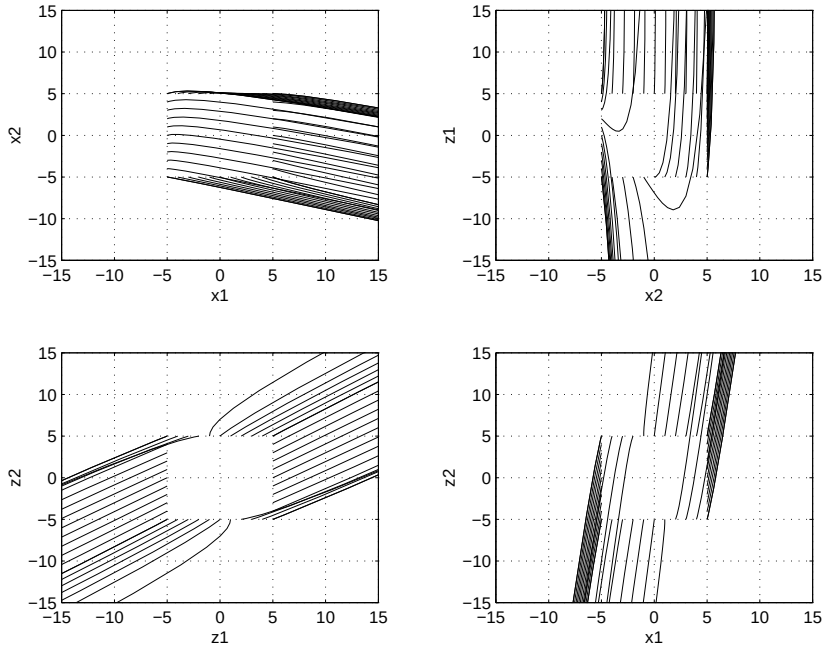


Figure 38: *State plane,* $\varepsilon = 0.1$.

When $\varepsilon = 1$ the state variables and the state plane are shown in Figure 39 and Figure 40 respectively.

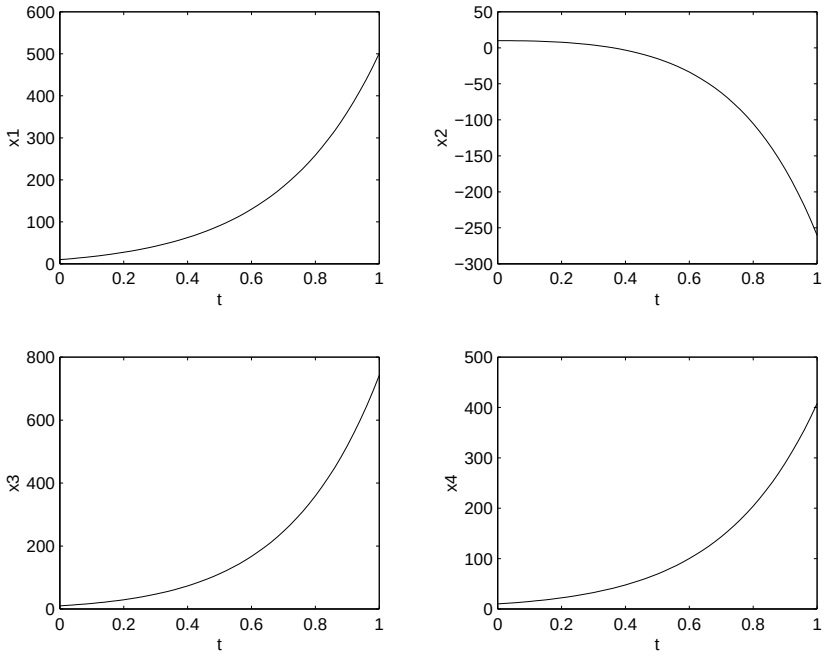


Figure 39: *State variable,* $\varepsilon = 1$.

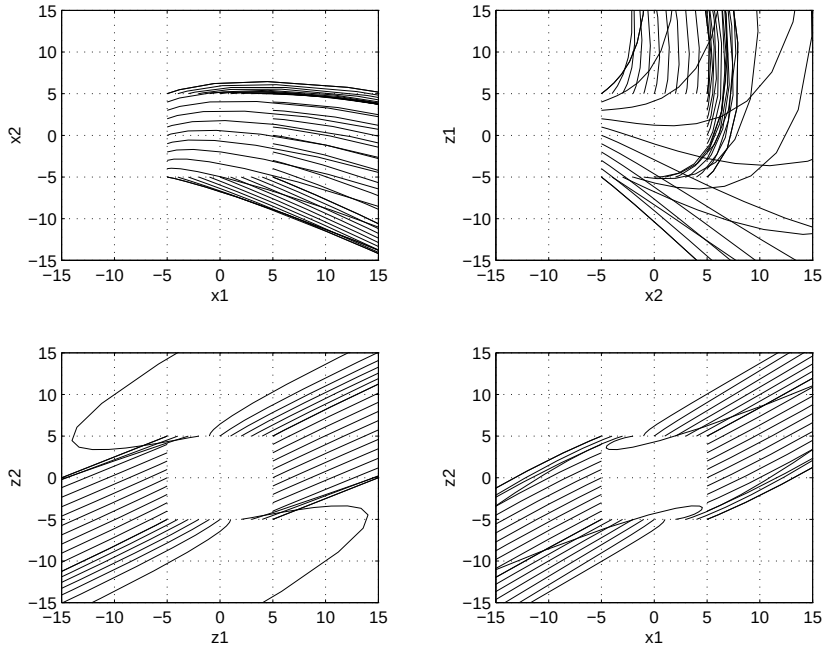


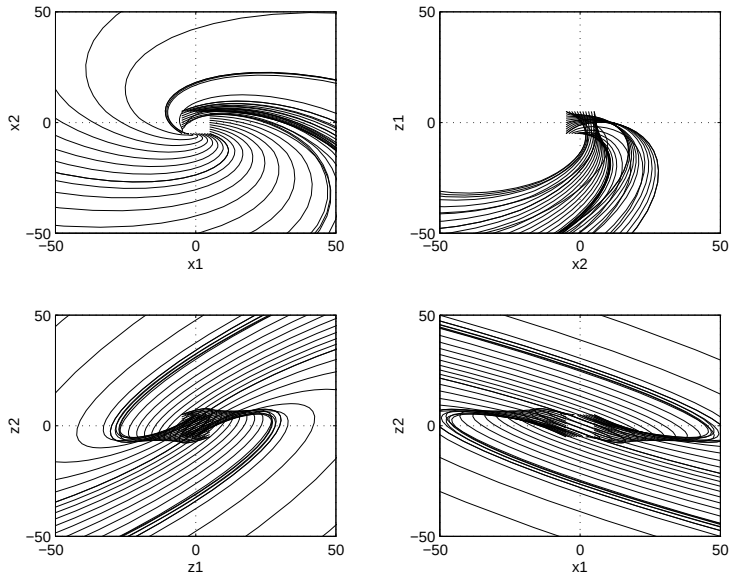
Figure 40: *State plane,* $\varepsilon = 1$.

With a sign function in feedback

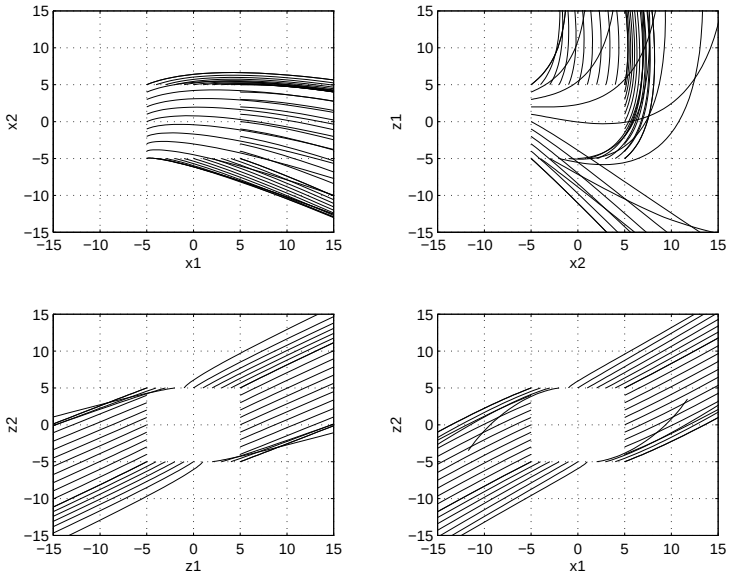
Next use a sign function of the state variables as feedbacks, ie.

$$u = \text{sgn}\,(x_1 + x_2 + z_1 + z_2)\,.$$

The state planes are shown in Figure 41. The response is unstable.



(a)



(b)

Figure 41: *State planes when input a sign function of state variables, (a) $\varepsilon = -1$, (b) $\varepsilon = 1$.*

In fact the input should have had a negative instead of a positive sign. Equation 65 and 66 turn into

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ \frac{2}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \frac{1}{\varepsilon} \\ 0 \end{bmatrix} \cdot \operatorname{sgn}\,(x_1 + x_2 + z_1 + z_2) \tag{67}$$

The new system has a structure which varies accordingly with the value of the sign function. Rewrite Equation 67 as

$$\dot{\Xi} = A\Xi + B \operatorname{sgn} \chi = \Upsilon\Xi, \tag{68}$$

where the switching hyperplane is represented by

$$\chi = x_1 + x_2 + z_1 + z_2$$

and the rest of the symbols are

$$\Xi = \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix},\ A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ \frac{2}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{\varepsilon} \\ 0 \end{bmatrix},$$

and

$$\Upsilon = \begin{bmatrix} 2 + \operatorname{sgn}\chi & 2 + \operatorname{sgn}\chi & 1 + \operatorname{sgn}\chi & 1 + \operatorname{sgn}\chi \\ -1 + \operatorname{sgn}\chi & 1 + \operatorname{sgn}\chi & -1 + \operatorname{sgn}\chi & 1 + \operatorname{sgn}\chi \\ \frac{2+\operatorname{sgn}\chi}{\varepsilon} & \frac{1+\operatorname{sgn}\chi}{\varepsilon} & \frac{2+\operatorname{sgn}\chi}{\varepsilon} & \frac{1+\operatorname{sgn}\chi}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} \end{bmatrix}.$$

The characteristic of the new system depends on eigenvalues of the matrix Υ which may take either one of these three values,

$$\Upsilon = \begin{cases} \Upsilon_1, & \chi > 0 \\ A, & \chi = 0 \\ \Upsilon_2, & \chi < 0 \end{cases}, \tag{69}$$

where

$$\Upsilon_1 = \begin{bmatrix} 3 & 3 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \frac{3}{\varepsilon} & \frac{2}{\varepsilon} & \frac{3}{\varepsilon} & \frac{2}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} \end{bmatrix}, \quad \text{and} \quad \Upsilon_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ \frac{1}{\varepsilon} & 0 & \frac{1}{\varepsilon} & 0 \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{1}{\varepsilon} & \frac{2}{\varepsilon} \end{bmatrix}.$$

When $\chi = 0$ the eigenvalues had been shown earlier in Figure 26. The rest two cases are shown in Figure 42. Figure 42 (a)-(c) are the case where $\chi > 0$ while Figure 42 (d)-(f) are the case where $\chi < 0$. Figure 42 (d) shows the system to be stable, in other words all the eigenvalues λ_i satisfies the condition

$$Re\,\lambda_i \leq 0,$$

if ε satisfies the condition

$$-1 < \varepsilon < -0.5, \quad \chi < 0.$$

When $\chi > 0$ the system is always unstable.

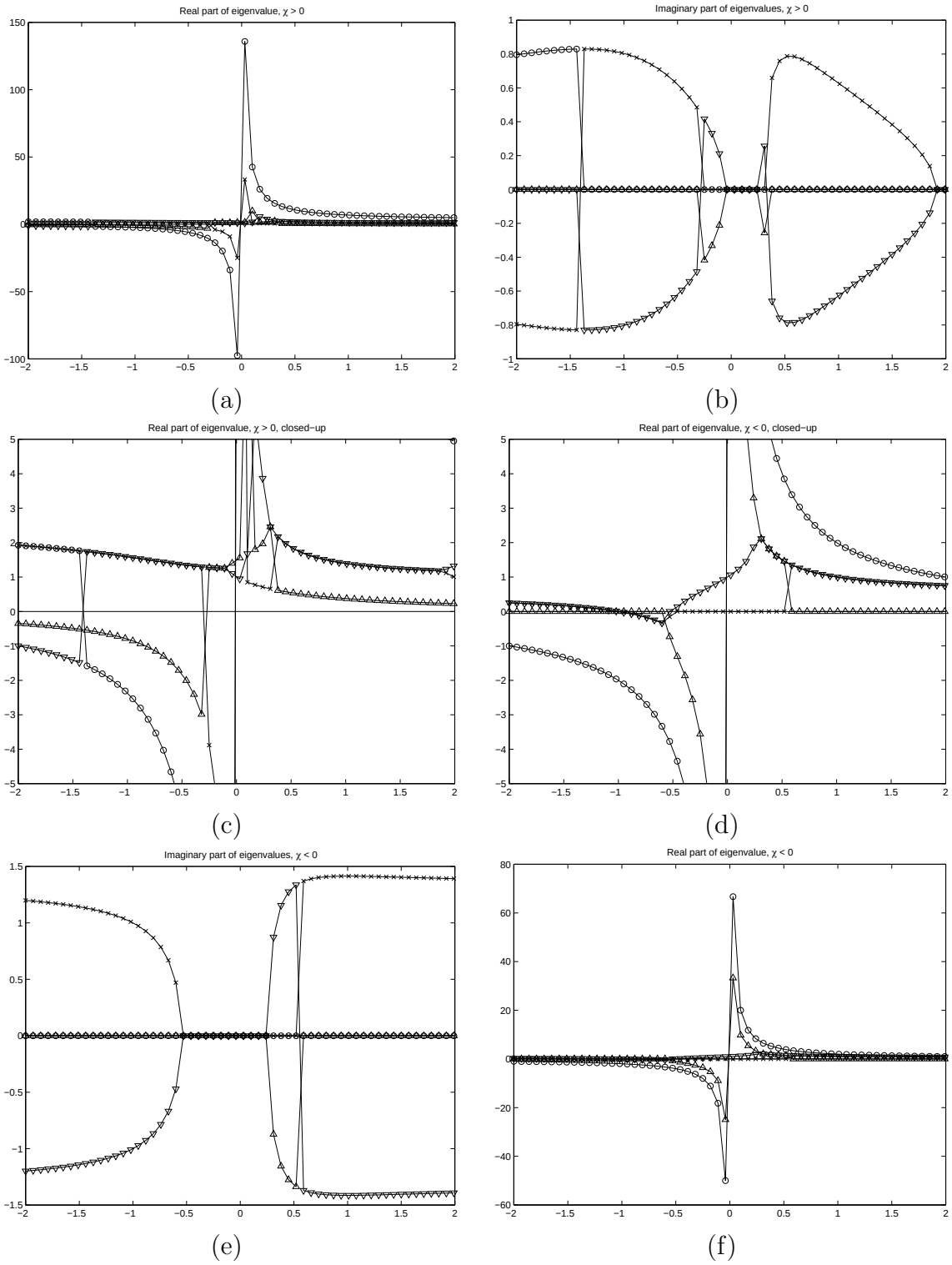

Real part of eigenvalue, χ > 0
(a)
Imaginary part of eigenvalues, χ > 0
(b)
Real part of eigenvalue, χ > 0, closed−up
(c)
Real part of eigenvalue, χ < 0, closed−up
(d)
Imaginary part of eigenvalues, χ < 0
(e)
Real part of eigenvalue, χ < 0
(f)

Figure 42: *Movement of eigenvalues with ε when $\chi > 0$ ((a)-(c)), and when $\chi < 0$ ((d)-(f))*

Variable structure feedback

Let

$$u = -K\text{sgn } s, \tag{70}$$

where

$$s = c_1x_1 + c_2x_2 + c_3z_1 + z_2 \tag{71}$$

is a switching hyperplane. It follows that

$$s = c_1\dot{x}_1 + c_2\dot{x}_2 + c_3\dot{z}_1 + \dot{z}_2 \tag{72}$$

The state equations Equation 65 and 66 become

$$\begin{aligned}
\dot{x}_1 &= 2x_1 + 2x_2 + z_1 + z_2 - K\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right) && (73)\\
\dot{x}_2 &= -x_1 + x_2 - z_1 + z_2 - K\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right) && (74)\\
\dot{z}_1 &= \frac{2}{\varepsilon}x_1 + \frac{x_2}{\varepsilon} + \frac{2}{\varepsilon}z_1 + \frac{z_2}{\varepsilon} - \frac{K}{\varepsilon}\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right) && (75)\\
\dot{z}_2 &= \frac{x_1}{\varepsilon} + \frac{x_2}{\varepsilon} + \frac{z_1}{\varepsilon} + \frac{2}{\varepsilon}z_2. && (76)
\end{aligned}$$

Let the Lyapunov function be

$$V = \frac{1}{2}s^2. \tag{77}$$

The derivative of the Lyapunov function is

$$\begin{aligned}
\dot{V} &= s\dot{s} \\
&= \left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right)\left[\frac{x_1}{\varepsilon}\frac{x_2}{\varepsilon} + \frac{z_1}{\varepsilon} + \frac{2}{\varepsilon}z_2 + \right. \\
&\quad c_2\left(-x_1 + x_2 - z_1 + z_2 - K\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right)\right) + \\
&\quad c_1\left(2x_1 + 2x_2 + z_1 + z_2 - K\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right)\right) + \\
&\quad \left. c_3\left(\frac{2}{\varepsilon}x_1 + \frac{x_2}{\varepsilon} + \frac{2}{\varepsilon}z_1 + \frac{z_2}{\varepsilon} - \frac{K}{\varepsilon}\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right)\right)\right] && (78)\\
&= \frac{1}{\varepsilon}\left[\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right)\left(\left(1 + 2c_3 + 2c_1\varepsilon - c_2\varepsilon\right)x_1 + \right.\right. \\
&\quad x_2 + c_3x_2 + 2c_1\varepsilon x_2 + c_2\varepsilon x_2 + z_1 + 2c_3z_1 + c_1\varepsilon z_1 - c_2\varepsilon z_1 + \\
&\quad \left.\left. 2z_2 + c_3z_2 + c_1\varepsilon z_2 + c_2\varepsilon z_2 - \left(c_3 + \left(c_1 + c_2\right)\varepsilon\right)K\text{sgn}\left(c_1x_1 + c_2x_2 + c_3z_1 + z_2\right)\right)\right] && (79)
\end{aligned}$$

The following figures plot derivative of the Lyapunov function with respect to x_1, x_2, z_1, and z_2. The points plotted follows points shown in the Initial Conditions Set 4. The

Initial Condition Set 4 `k = 0, k is the time step`

```
for p = -5, -4, ..., 5
  for q = -5, -4, ..., 5
    for r = -5, -4, ..., 5
      for s = -5, -4, ..., 5
        x(k+1) = (p, q, r, s);
      endfor;
    endfor;
  endfor;
endfor;
```

The parameters were

$$c_1 = 1.7, \quad c_2 = 0.5, \quad c_3 = 4.3.$$

The feedback gain K is 35 for Figure 43, 44, 45, and 46. These four figures are projections of a four dimentional hyper-space onto each of the four axes.
Figure 43 shows the plot when $\varepsilon = 2$. The system is stable since

$$\dot{V} < 0 \; \forall x \in \mathcal{D}^4 \subset \mathcal{R}^4, \quad -5 < \mathcal{D} < 5, \; \mathcal{D} \subset \mathcal{R}.$$

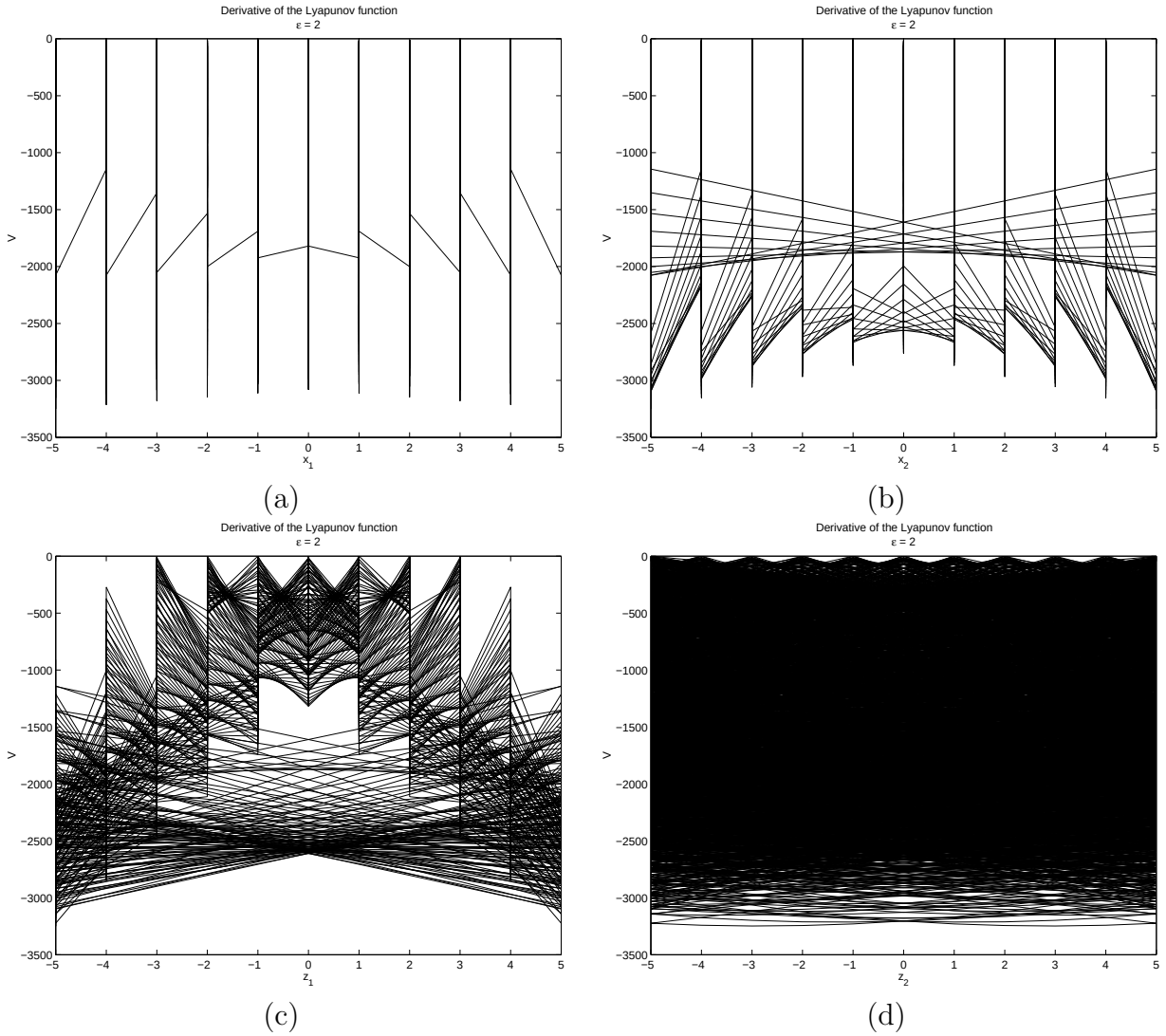


Figure 43: *Derivative of Lyapunov function (4^{th}-order singularly perturbed system), $\varepsilon = 2$*

When $\varepsilon = 0.1$ the result is shown in Figure 44. A closed-up of Figure 44 shows that the system is unstable since $\exists x$ such that $V \approx 350 > 0$.

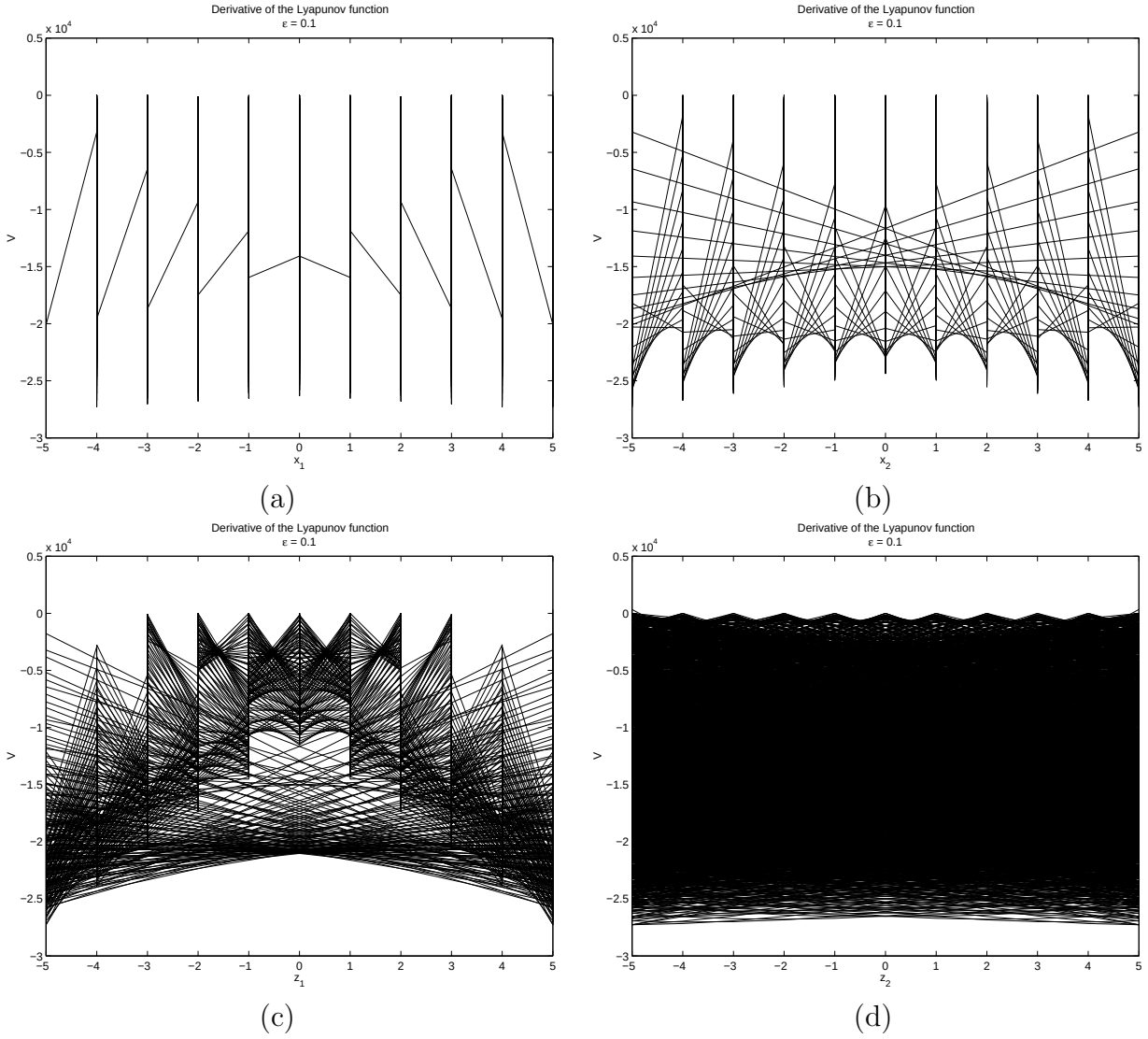


Figure 44: *Derivative of Lyapunov function (4^{th}-order singularly perturbed system), $\varepsilon = 0.1$*

Stability of a system can be determined by using either

1. the Lyapunov's stability theorem ([Kha96], Theorem 3.1) or
2. the Lyapunov's indirect method ([Kha96], Theorem 3.7.)

The first one is based on the Lyapunov's function $V(x)$ while the second one on whether the Jacobian matrix $A = \frac{\partial f(x)}{\partial x}$ evaluated at the equilibrium point is a stability matrix. Both method has got its own unique , namely

(to be continued on page 59)

For $\varepsilon < 0$ similar plots can be shown as the plots of Figure 45 and 46 Figure 45 shows that the result when $\varepsilon = -1$ is unstable.

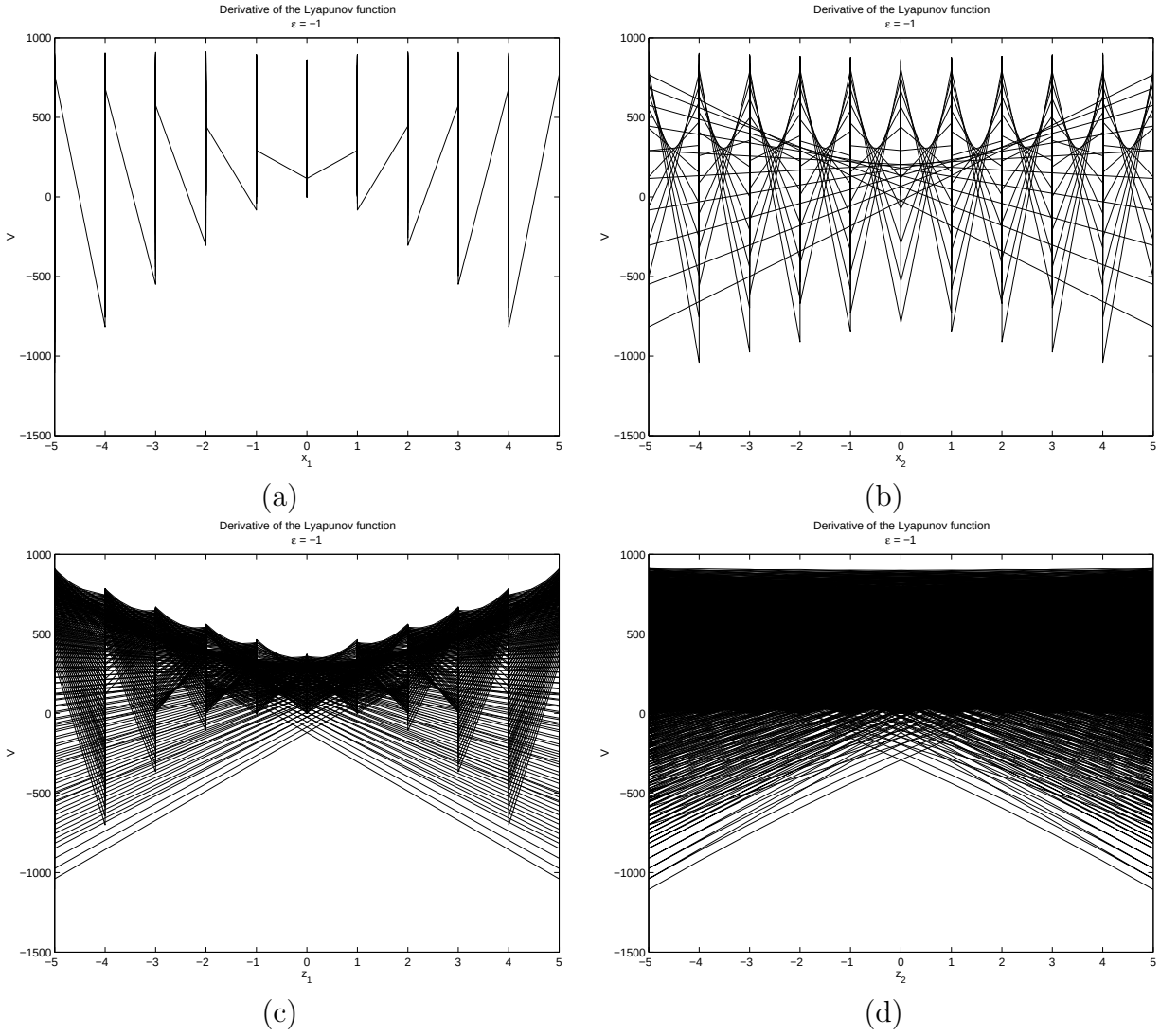


Figure 45: *Derivative of Lyapunov function (4^{th}-order singularly perturbed system), $\varepsilon = -1$*

(Continued from page 58, about stability)

1. the 's stability theorem only provides a sufficient condition for stability; failure of one Lyapunov function candidate does not imply failure of another Lyapunov function (therefore if no successful candidate can be found then nothing may be said about the stability or the asymptotic stability of the system),

2. the Lyapunov's indirect method does not give any about the stability of a system if

$$\operatorname{Re}\lambda_i \leq 0 \quad \forall i \qquad \text{and} \qquad \exists i \text{ such that } \operatorname{Re}\lambda_i = 0.$$

(to be continued on page 60)

Figure 46 shows the result when $\varepsilon = -0.01$. The result shows that the system is also unstable for a smaller in magnitude of $\varepsilon < 0$.

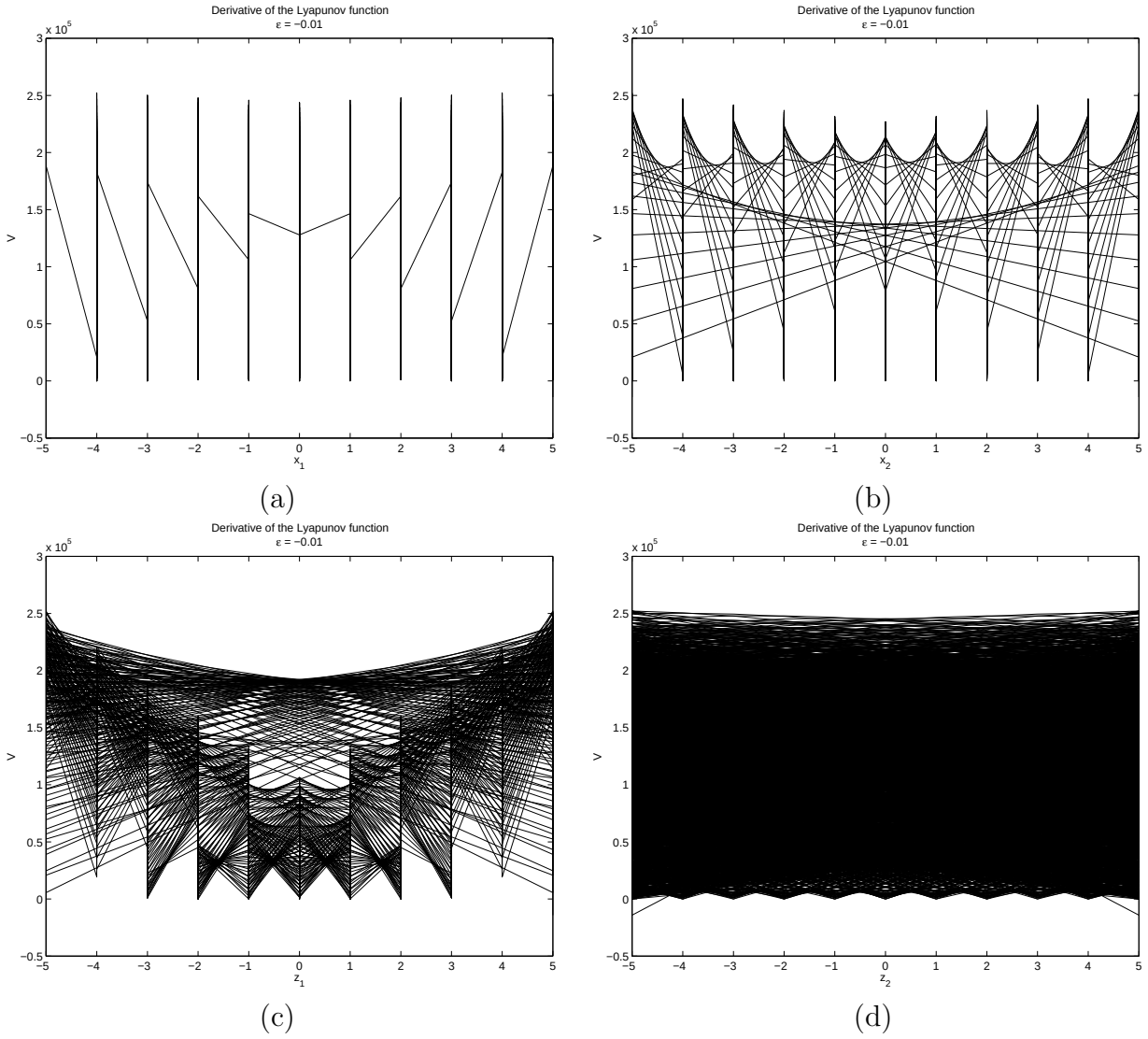


Figure 46: *Derivative of Lyapunov function (4^{th}-order singularly perturbed system), $\varepsilon = -0.01$*

(Continued from page 59, about stability)

The Lyapunov

$$PA + A^T P = -Q, \quad Q \text{ positive definite} \tag{80}$$

is the link between the two method. In order to find out whether A is a stability matrix or not, one can either

1. find all the eigenvalues λ_i of A, if $\mathrm{Re}\,\lambda_i < 0$ then A is a stability matrix, or
2. solve Equation 80, if it has a definite solution P then A is a stability matrix.

(to be continued on page 61)

Next increase the value of the gain K. In comparison to Figure 44 increase the gain K ten times (ie K is now 350) and show the result in Figure 47. From Figure 47 the system became stable after the gain was increased.

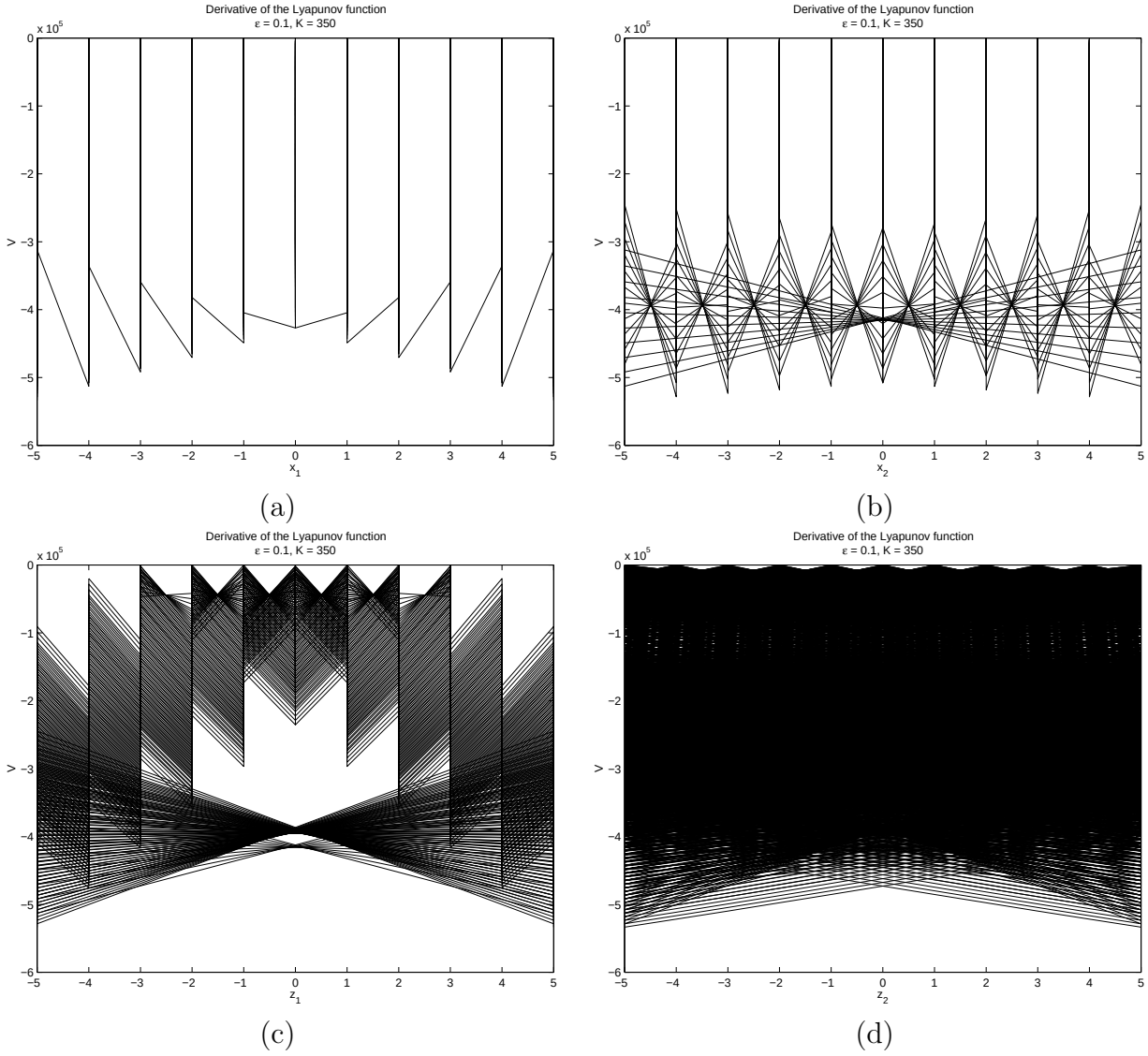


Figure 47: *Derivative of Lyapunov function (4^{th}-order singularly perturbed system), $\varepsilon = 0.1$, K was increased ten times.*

(Continued from page 60, about stability)

Which method is better depends on whether there is any uncertainty in the system . That is,

1. if A is a constant matrix, then finding the eigenvalues is better recommended since λ_i's also give direct information about the response of the linearized system.

2. if A is a perturbed matrix, then finding the solution of the Lyapunov equation is necessary because its solution leads to a Lyapunov function, the existence of which allows conclusions about the stability of the system. ♢

However in the case when $\varepsilon = -1 < 0$ increasing the gain K ten times did not make the stable stable, compare Figure 48 with Figure 45.

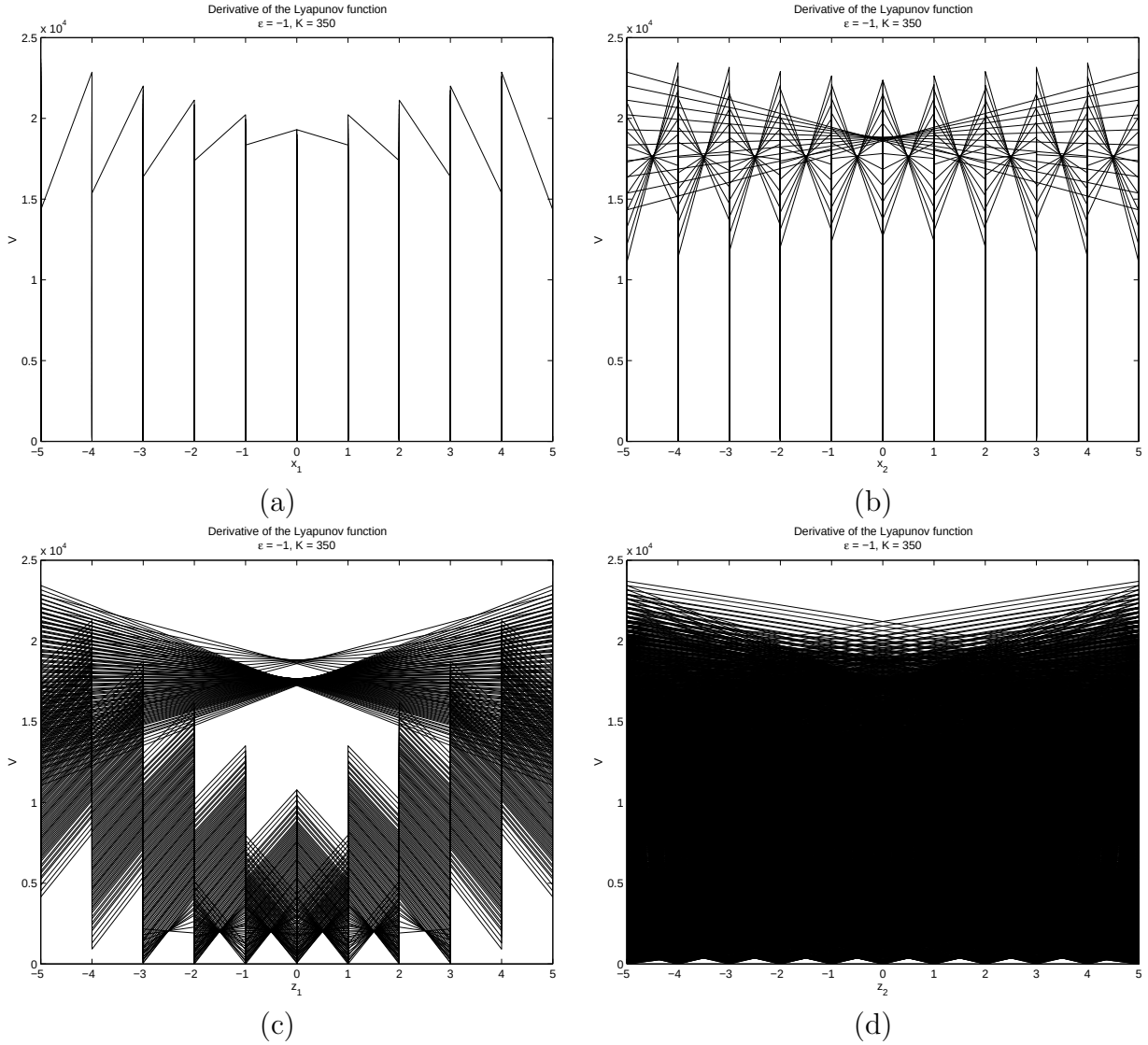


Figure 48: *Derivative of Lyapunov function (4^{th}-order singularly perturbed system), $\varepsilon = -1$, K was increased ten times.*

When a Lyapunov function $V(x)$ satisfies the conditions

$$V(0) = 0,\, V(x) > 0\ \forall x \in \mathcal{D} - 0,\ \text{and}\ (\dot{V})(x) \leq 0 \quad \forall x \in \mathcal{D}, \quad \mathcal{D} \subset \mathcal{R}^n,$$

$x = 0$ being the equilibrium point, it does not mean that $\mathcal{D}$ is the domain of attraction for the system because there is no guarantee that a trajectory which starts from $x(0) \in \mathcal{D}$ will remain forever in $\mathcal{D}$. It only means that the equilibrium point is stable and, if furthermore

$$\dot{V}(x) < 0, \quad \forall x \in \mathcal{D} - 0$$

then the equilibrium point is asymptotically stable. The domain on which the Lyapunov function is valid has got nothing to do with the domain of attraction of the system. ♡

Phase plane plots of the system

A couple of phase plane plots are shown for

1. $\varepsilon = 2,\ K = 35$ (Figure 49 (a) and (b)) which is stable,
2. $\varepsilon = 0.1,\ K = 35$ (Figure 49 (c) and (d)) which is unstable except perhaps within a limitted domain of attraction,
3. $\varepsilon = 0.1,\ K = 350$ (Figure 49 (e) and (f)) which is stable.

The parameters were

$$c_1 = 1.7, \quad c_2 = 0.5, \quad c_3 = 4.3.$$

And the initial conditions used were the Initial Condition Set 5.

Initial Condition Set 5 *The following set points*

- *five points on the line connecting* $(-7, -6, -4, -13)$ *and* $(8, 5, 12, 3)$*, namely* $(-7, -6, -4, -13)$*,* $(-4, -3.8, -0.8, -9.8)$*,* $(-1, -1.6, 2.4, -6.6)$*,* $(2, 0.6, 5.6, -3.4)$*, and* $(5, 2.8, 8.8, -0.2)$
- *five points on the line connecting* $(14, -7, 5, -5)$ *and* $(-3, 5, -10, 4)$*, namely* $(-3, -7, -10, -5)$*,* $(0.4, -4.6, -7, -3.2)$*,* $(3.8, -2.2, -4, -1.4)$*,* $(7.2, 0.2, -1, 0.4)$*, and* $(10.6, 2.6, 2, 2.2)$
- *five points on the line connecting* $(-6, 13, 4, -3)$ *and* $(21, -3, -4, 8)$*, namely* $(-6, -3, -4, -3)$*,* $(-0.6, 0.2, -2.4, -0.8)$*,* $(4.8, 3.4, -0.8, 1.4)$*,* $(10.2, 6.6, 0.8, 3.6)$*, and* $(15.6, 9.8, 2.4, 5.8)$.
- *five points on the line connecting* $(-4, 6, 5, 4)$ *and* $(5, -3, -5, -2)$*, namely* $(-4, -3, -5, -2)$*,* $(-2.2, -1.2, -3, -0.8)$*,* $(-0.4, 0.6, -1, 0.4)$*,* $(1.4, 2.4, 1, 1.6)$*, and* $(3.2, 4.2, 3, 2.8)$.
- *five points on the line connecting* $(-3, -8, 3, -1)$ *and* $(14, 2, -1, 5)$*, namely* $(-3, -8, -1, -1)$*,* $(0.4, -6, -0.2, 0.2)$*,* $(3.8, -4.0, 0.6, 1.4)$*,* $(7.2, -2, 1.4, 2.6)$*, and* $(10.6, 0, 2.2, 3.8)$.
- *five points on the line connecting* $(-10, 7, -8, -12)$ *and* $(13, -9.3, 7, 1.2)$*, namely* $(-10, -9.3, -8, -12)$*,* $(-5.4, -6.04, -5, -9.36)$*,* $(-0.8, -2.78, -2, -6.72)$*,* $(3.8, 0.48, 1, -4.08)$*, and* $(8.4, 3.74, 4, -1.44)$.

Figure 49 (a)-(f) show these plots. Comparing Figure 49 (c)-(d) with Figure 49 (a)-(b), when *varepsilon* becomes smaller the system becomes less stable. Note that the plot of the Lyapunov's derivative (Figure 43 and Figure 44) also shows that $\varepsilon = 0.1$ is less stable than $\varepsilon = 2$ since Figure 43 is negative-semidefinite while Figure 44 is neither negative-definite nor negative-semidefinite. The stable region of Figure 49 (a) and (b) ($\varepsilon = 2$) is larger than that of Figure 49 (c)-(d) ($\varepsilon = 0.1$.) When the feedback gain K was increased (Figure 49 (e)-(f)) the system becomes more stable. The simulation time was $t \leq 3$sec.

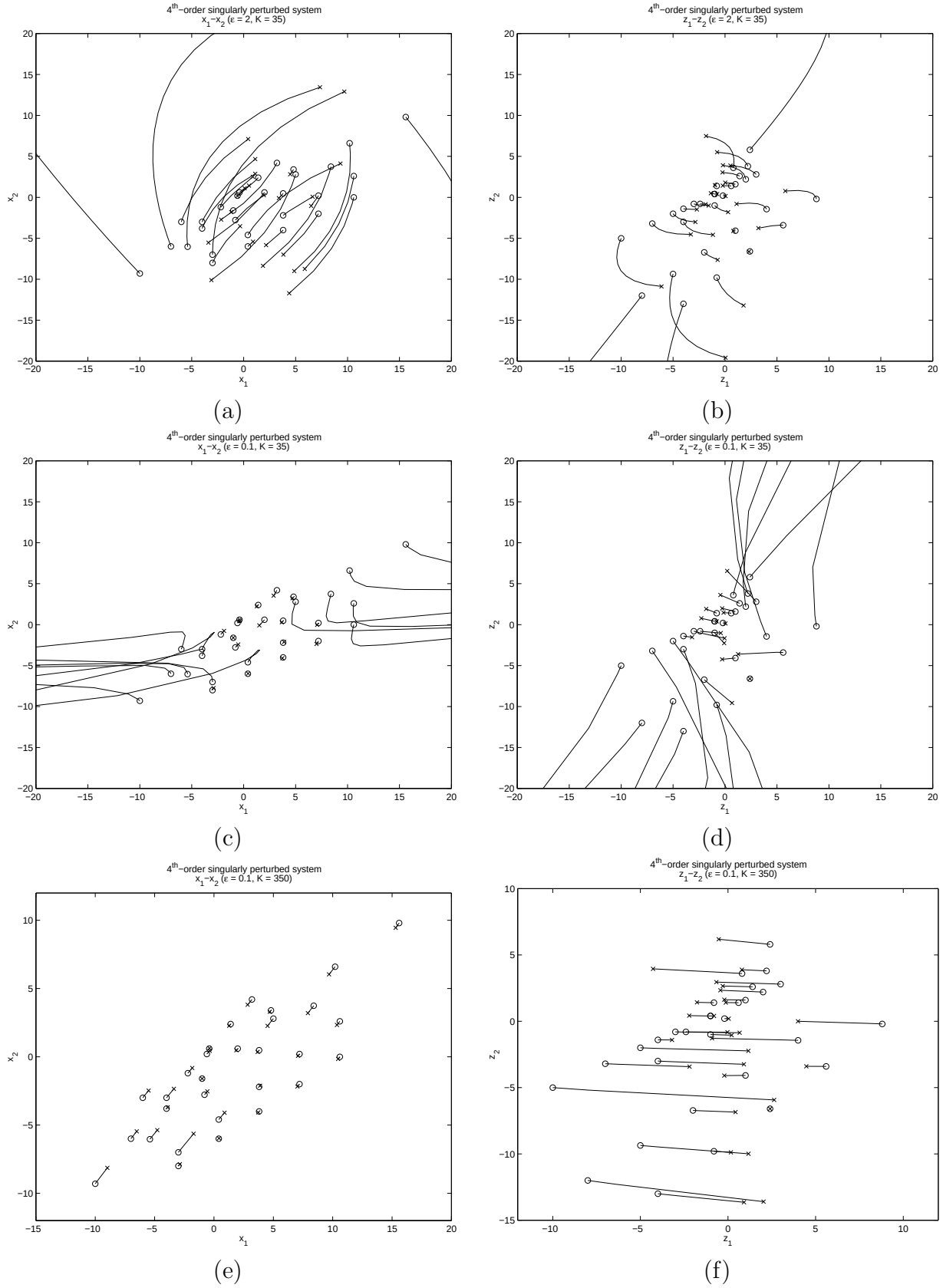


Figure 49: *Phase planes (4^{th}-order singularly perturbed system) (a)-(b)* $\varepsilon = 2, K = 35$*, (c)-(d)* $\varepsilon = 0.1, K = 35$*, (e)-(f)* $\varepsilon = 0.1, K = 350$

A polygonal phase plane

This section is an attempt to find a way of displaying movement of more than three state variables on a two dimensional plane. This method is named a *polygonal phase plane* has the following description.

1. The axes for the state variables are represented by lines going outward from the origin. The axes are equally spaced around the origin.

2. Each axes has a normallized scale from $r = 0.1$ for the minimum value of the simulated state variable to $r = 1$ for the maximum value, r is the radius starting from the origin.

3. Plot the state variables as snapshots of polygons each having a vertex on each axis. The number of snapshots is ≤ 10. Also plot the starting point as a circle and the end point as a triangle.

4. The order of the axes starts from the one which goes toward the north as the first one or x_1 with the the other state variables having an axis in a direction on a line rotated clockwise.

5. Whether the system converges or not can be seen from the simulated numerical data. If the data is still varying with each step at end of the simulation time, then it is probable that it diverges.

Figure 50 (a) to (f) show polygonal phase plots of the system. Following is the description of each subfigure.

Figure 50 (a) is when the parameters were

$$\varepsilon = 0.01,\ c_1 = 3.7,\ c_2 = 1.5,\ c_3 = 0.3,\ K = 500.$$

The initial conditions was $(7, -8, 4, 3)$. The range of the axes are normalized with the original values of $[6.742, 7]$ for x_1, $[-8.2553, -8]$ for x_2, $[-22.2527, 4]$ for z_1, and $[2.7901, 3.0648]$ for z_2. The simulation time was less than 3 seconds. The system is stable and the result ends on a hyperplane.

Figure 50 (b) is when the parameters were

$$\varepsilon = 0.1,\ c_1 = 0.7,\ c_2 = 1.5,\ c_3 = 2.3,\ K = 50.$$

The initial conditions was $(3, -4, 6, 0)$. The range of the axes are normalized with the original values of $[2.8527, 3]$ for x_1, $[-4.1956, -4]$ for x_2, $[4.6955, 6]$ for z_1, and $[0, 0.1358]$ for z_2. The simulation time was less than 3 seconds. The system is stable and the result ends on a hyperplane.

Figure 50 (c) is when the parameters were

$$\varepsilon = 1,\ c_1 = 2.4,\ c_2 = 1.2,\ c_3 = 2,\ K = 5.$$

The initial conditions was $(5, 1, 3, 2)$. The range of the axes are normalized with the original values of $[5, 1.1 \times 10^5]$ for x_1, $[-49950, 0]$ for x_2, $[3, 1.51 \times 10^5]$ for z_1, and $[2, 1.25 \times 10^5]$ for z_2. The simulation time was less than 3 seconds. The system is not stable and the result did not converge.

Figure 50 (d) is when the parameters were

$$\varepsilon = -0.1,\ c_1 = 1.1,\ c_2 = 0.8,\ c_3 = 2.1,\ K = 32.$$

The initial conditions was $(-7, -2, -6, 1)$. The range of the axes are normalized with the original values of $[-7, 6.37 \times 10^3]$ for x_1, $[-2, 4.63 \times 10^3]$ for x_2, $[-9.89 \times 10^3, -6]$ for z_1, and $[-510, 7.4]$ for z_2. The simulation time was 3 seconds. The result did not converge therefore the system is unstable.

Figure 50 (e) is when the parameters were

$$\varepsilon = 1,\ c_1 = 1.7,\ c_2 = 1.2,\ c_3 = 2.3,\ K = 400.$$

The initial conditions was $(0, 4, 12, 7)$. The range of the axes are normalized with the original values of $[-4.03, 0]$ for x_1, $[-0.199, 4]$ for x_2, $[8.08, 12]$ for z_1, and $[7, 7.26]$ for z_2. The simulation time was less than 3 seconds. The system converges. This can be compared with Figure 50 (c) which also has $\varepsilon = 1$ but which has a smaller value of K.

Figure 50 (f) is when the parameters were

$$\varepsilon = 0.001,\ c_1 = 1.9,\ c_2 = 1.3,\ c_3 = 2.8,\ K = 400.$$

The initial conditions was $(-5, 0.3, 7, -3)$. The range of the axes are normalized with the original values of $[-5.0014, -5]$ for x_1, $[0.2986, 0.3]$ for x_2, $[5.5883, 7]$ for z_1, and $[-3.0156, -3]$ for z_2. The simulated time was less than 3 seconds. Because there are very few snapshots in the result, one knows that the system converged quickly onto a hyperplane.

The following idea is new, and has yet got to be formally realized. I will do this in the course of the making the subsequent technical reports. The brief outline is drawn as follows.

1. A general polygonal phase plane for high dimension system may be drawn (in similar manner to the results given above) by increasing the number of axis. Generally speaking, if a system is n-dimensional with x_i, $i = 1, 2, \ldots, n$ as its state variables, then the angles these state variables make clock-wise with respect to the vertical 0-o'clock axis are

$$\theta_k = \frac{2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n.$$

2. A *normalized* polygonal phase plane may be drawn with all the axis normalized to the nominal value of the state variable it represents. This idea of normalized values is used extensively in power system engineering. One possible drawback could be the fact that, unlike in the field of power system engineering where the parameters does not vary dramatically away from their nominal values, the state variables in control system engineering could fluctuate with virtually no limit.

3. With the aid of computer softwares the polygonal phase plane plots could be put together to form a movie showing a continuous movement of the *trajectory.*

4. In order to distinguish the new trajectory in an n-dimensional polygonal phase space from the trajectory of a conventional state plane, a new terminology adopted from other existing disciplines of science *metamorphosis* shall be used to describe the movement of the *phase polygon* shown by these snapshots. This terminology (ie *phase-metamorphosis*) will be better equipped to accommodate a video sequence (a it phase-metamorphosis video) made from these snapshots (a *phase-metamorphosis diagram.*)

5. Lastly, (conventional) phase planes (2-dimensional,) phase spaces (3-dimensional) as well as the time-domain plots of the state variables may be incorporated into polygonal phase-metamorphosis diagram as small separate boxes.

This outline will be improved upon in other technical reports.

∞

The degenerate system

Then let $\varepsilon = 0$ and obtain a degenerate system

$$\begin{array}{rclcrcl} \dot{x^0} & = & A_1 x^0 + B_1^0 u, & & x^0(0) & = & x(0) \\ z^0 & = & A_2 x^0 + B_2^0 u, & & z^0(0) & \neq & z(0) \end{array} \tag{81}$$

where

$$A_1 = A_{11} - A_{12} A_{22}^{-1} A_{21} \tag{82}$$
$$B_1^0 = B_1 - A_{12} A_{22}^{-1} B_2 \tag{83}$$
$$A_2 = -A_{22}^{-1} A_{21} \tag{84}$$
$$B_2^0 = -A_{22}^{-1} B_2 \tag{85}$$

The matrix A_{22} is assumed to be nonsingular. The effect of degeneration is two folds, ie

1. reduce the state variable z from its status as state variables

2. desert the initial condition $z(0)$ of z.

From the values of A_{11}, A_{12}, B_1, A_{21}, A_{22} and B_2 above it follows that $A_1 = \begin{bmatrix} 1 & 1.3333 \\ 0 & 1 \end{bmatrix}$, $B_1^0 = \begin{bmatrix} 0.6667 \\ 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -0.3333 \\ 0 & -0,3333 \end{bmatrix}$ and $B_2^0 = \begin{bmatrix} -0.6667 \\ 0.3333 \end{bmatrix}$.
The system reduces in dimension from 4^{th}-order to 2^{nd}-order. The new system is unstable since the two eigenvalues are both 1's.
A descriptive diagram comparing the structure of the original equations (Equation 65 and 66) with that of the degenerated system (Equation 81) can be seen in reference ([Nai88], Figure 2.3.)

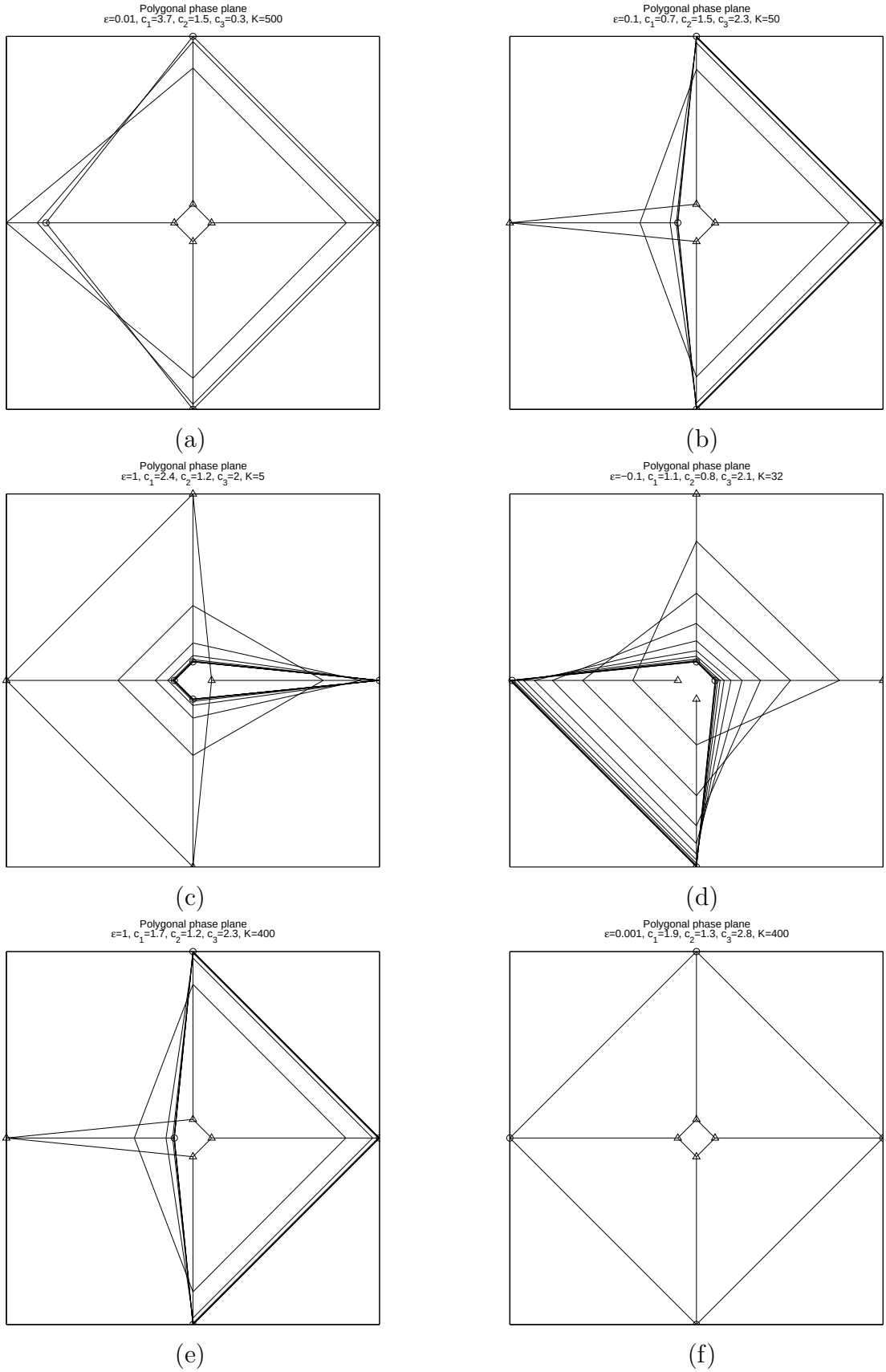


Figure 50: *Polygonal phase planes (a)* $\varepsilon = 0.01$*, (b)* $\varepsilon = 0.1$*, (c)* $\varepsilon = 1$*, (d)* $\varepsilon = -0.1$*, (e)* $\varepsilon = 1$ *(with a higher gain), and (f)* $\varepsilon = 0.001$

The state variables and the state plane are shown in Figure 51 and Figure 52 respectively. (**Errata:** The labels x_1, x_2, x_3 and x_4 written on the plots are x_1^0, x_2^0, z_1^0 and z_2^0 respectively.)

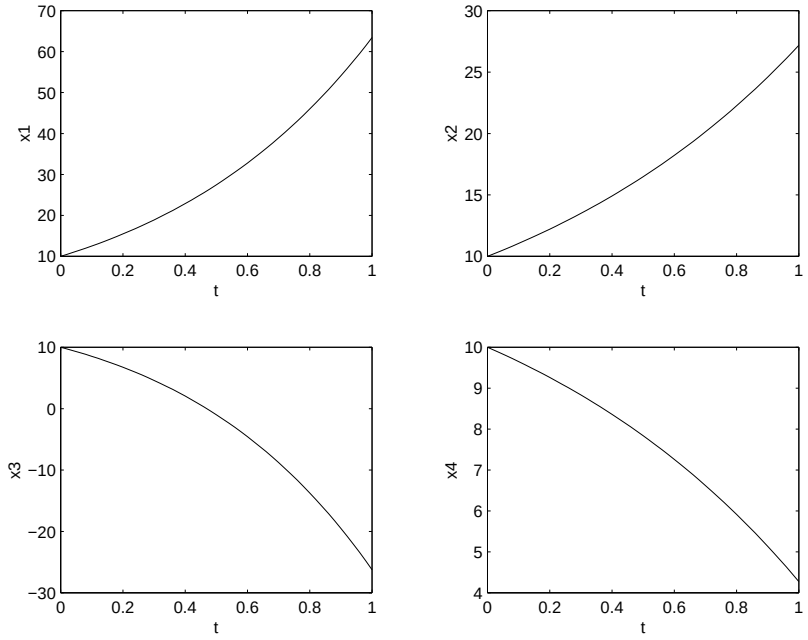


Figure 51: *State variable, degenerated system.*

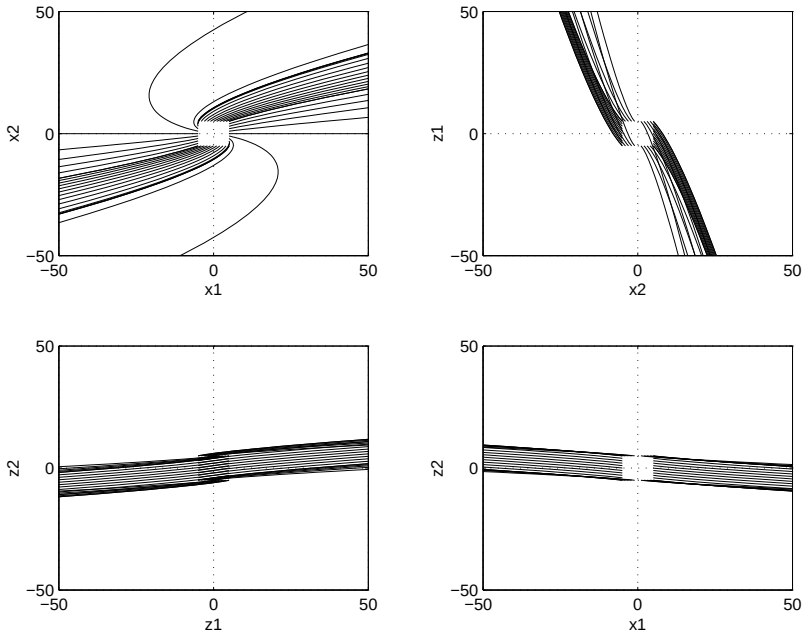
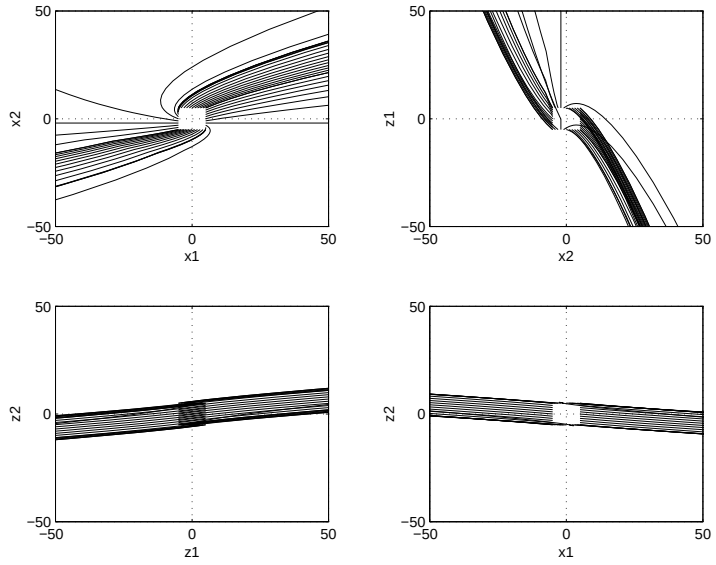


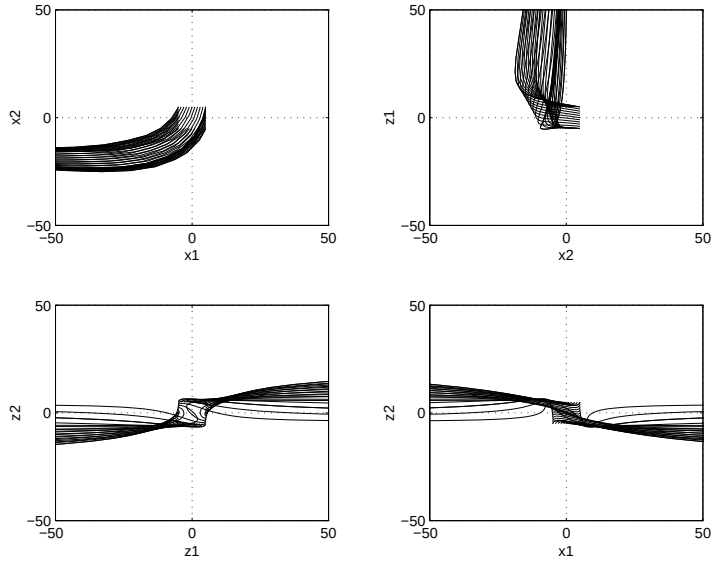
Figure 52: *State plane, degenerated system.*

The degenerate system with a sign function feedback

Input a sign function of state variables to the degenerated system and obtain Figure 53 (a). Add state feedbacks as well as the sign function input and obtain Figure 53 (b). (these simulations may be incorrect due to error in programming, I think it might have been done on a 4^{th}-order system by mistake)



(a)



(c)

Figure 53: *State plane, degenerated system with (a)* $u = \text{sgn}\,(x_1 + x_2 + z_1 + z_2)$, *(b)* $u = \text{sgn}\,(x_1 + x_2 + z_1 + z_2)$

Degenerated system with variable structure control

In this section the origin of the degenerated system was

1. stabilized via state feedback having a gain obtained from the solution of the Riccati equation described in page 19. This amounts to Equation 86 with $K = 0$. The result is shown in Figure 55 (a), then

2. a variable structure was then applied by using $K \neq 0$ in Equation 86. Figure 55 (b)-(f) show the response. In Figure 55 (f) Equation 87 with $\gamma = 10$ was used instead of Equation 86.

When simulating with a negative feedback of sign function ($K < 0$) the numerical iteration suffers indefinite repetition once the trajectory reaches the surface of the hyperplane. Therefore with negative feedback Equation 87 was used as an approximate of Equation 86 in simulating. An interesting observation when simulating with $K < 0, \gamma = 10$ in Equation 87 is that when $c \geq 0.99$ the trajectory converges toward the origin (eg. see Figure 54 (a),) but when $c < 0.98$ the system converges to two equilibrium points which are away from the origin (eg. see Figure 54 (b).)
The control input can be written in a general form as

$$u = -2.8229\, x_1 - 1.822 x_2 + K \,\mathrm{sgn}\,(c x_1 + x_2)\,, \tag{86}$$

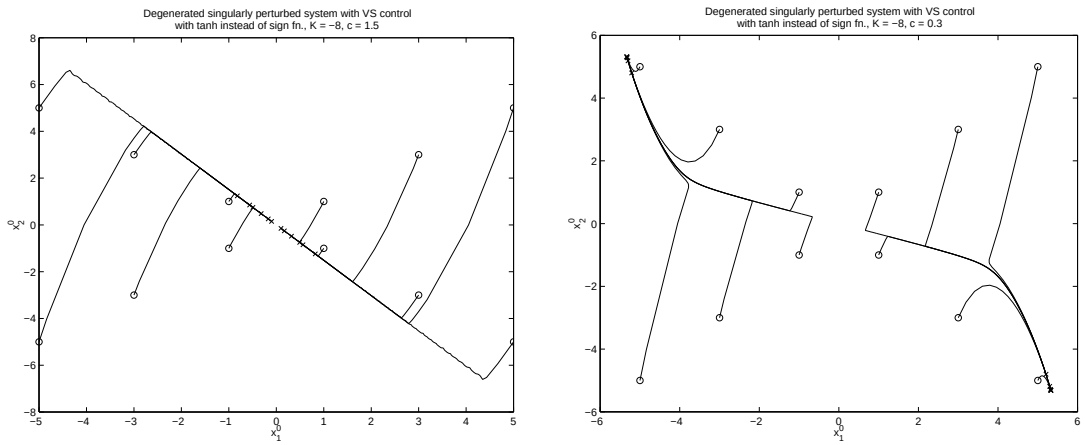
and a special control used as the approximate for Equation 86

$$u = -2.8229\, x_1 - 1.822 x_2 + K \,\tanh\,\gamma\,(c x_1 + x_2)\,. \tag{87}$$

Solving Equation 42 (a Riccati equation) for this system leads to

$$P = \begin{bmatrix} 3.4844 & 0.2499 \\ 0.2499 & 0.8281 \end{bmatrix},\ \Gamma = \begin{bmatrix} -1.4304 \\ -2.0972 \end{bmatrix},\ G = \begin{bmatrix} 2.8229 & 1.8228 \end{bmatrix},$$

where Γ is a matrix containing the closed-loop eigenvalues and G is the gain matrix. The Frobenius norm of the relative residual matrix is 6.836×10^{-15}..


Degenerated singularly perturbed system with VS control
with tanh instead of sign fn., K = −8, c = 1.5
Degenerated singularly perturbed system with VS control
with tanh instead of sign fn., K = −8, c = 0.3

(a) (b)

Figure 54: *Phase plane (a)* $K = -8, c = 1.5$ *(b)* $K = -8, c = 0.3$

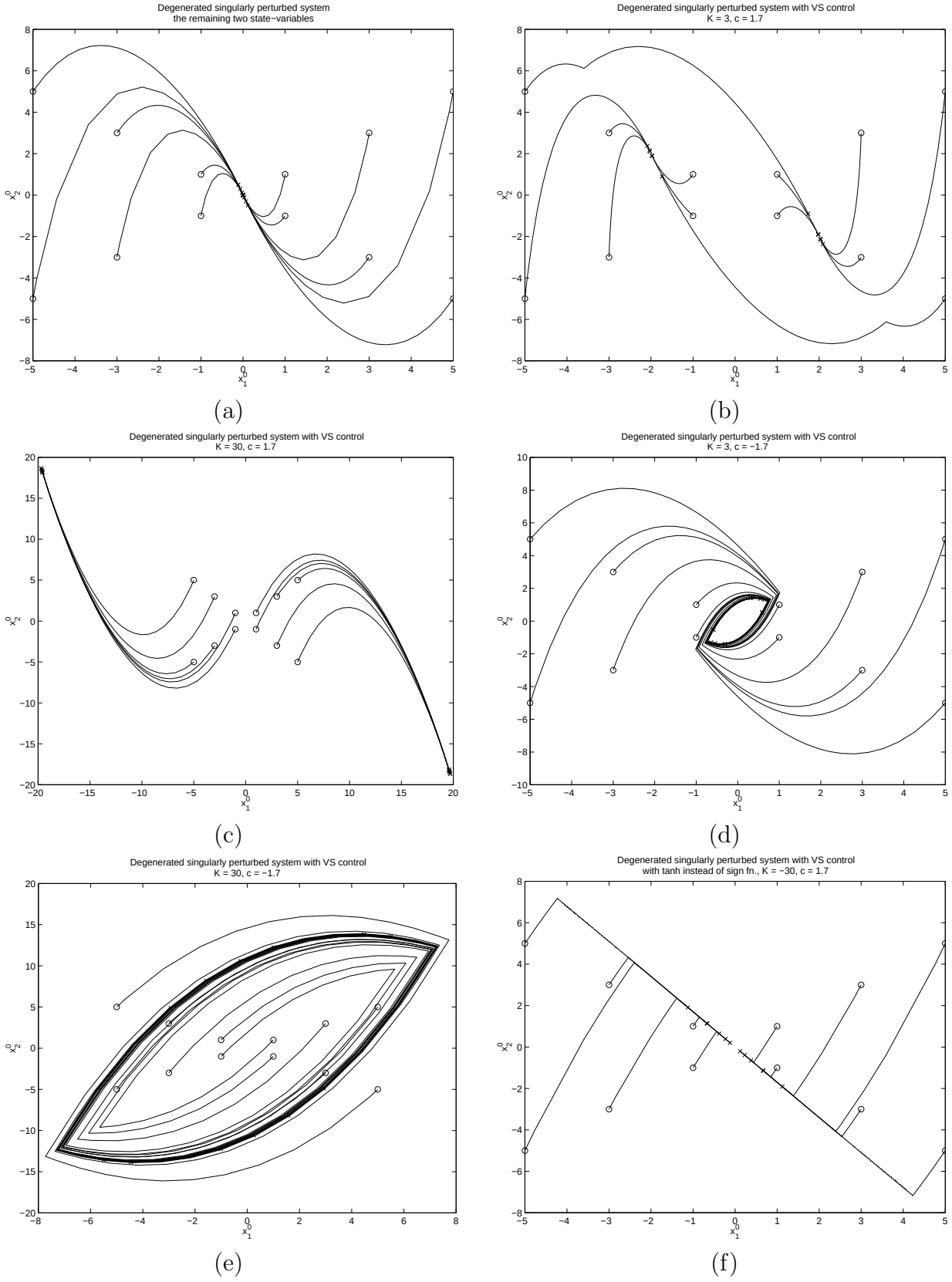

Degenerated singularly perturbed system
the remaining two state−variables
(a)
Degenerated singularly perturbed system with VS control
K = 3, c = 1.7
(b)
Degenerated singularly perturbed system with VS control
K = 30, c = 1.7
(c)
Degenerated singularly perturbed system with VS control
K = 3, c = −1.7
(d)
Degenerated singularly perturbed system with VS control
K = 30, c = −1.7
(e)
Degenerated singularly perturbed system with VS control
with tanh instead of sign fn., K = −30, c = 1.7
(f)

Figure 55: *Phase plane (a)* $K = 0$*, (b)* $K = 3, c = 1.7$*, (c)* $K = 30, c = 1.7$*, (d)* $K = 3, c = -1.7$*, (e)* $K = 30, c = -1.7$*, and (f)* $K = -30, c = 1.7$

Boundary value problem

29^{th} **April 1998**
Consider a boundary value problem [Nai88]

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{88}$$

with boundary conditions

$$x(0) = 5; \quad x(1) = 1. \tag{89}$$

The solution of this equation is

$$x = \frac{e^{\frac{1}{\varepsilon}} - 5}{e^{\frac{1}{\varepsilon}} - 1} + 4\frac{e^{\frac{1}{\varepsilon}} e^{\frac{-t}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - 1} \tag{90}$$

was shown in Figure 56. Figure 56 (d) shows the process of degeneration (ie. changing from $\varepsilon \neq 0$ to $\varepsilon = 0$) which is opposite to the process of singular perturbation (ie. changing from $\varepsilon = 0$ to $\varepsilon \neq 0$) and which is characterized by the loss of some boundary conditions. Figure 56 (g) is the combination of Figure 56 (a) to (f).

Another boundary value problem

Next consider another boundary value problem [Nai88]

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0 \tag{91}$$

with boundary conditions

$$x(0) = 13, \quad x(1) = 3. \tag{92}$$

The solution of this system is

$$\begin{aligned} x \;=\; & \left(-13\, e^{1/2\,\frac{1+\sqrt{1-4\,\varepsilon}}{\varepsilon}} e^{1/2\,\frac{-1+\sqrt{1-4\,\varepsilon}}{\varepsilon}} + 3\, e^{1/2\,\frac{1+\sqrt{1-4\,\varepsilon}}{\varepsilon}} + 13\, \left(e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon}}\right)^2 - 13\right) \\ & e^{1/2\,\frac{(-1+\sqrt{1-4\,\varepsilon})t}{\varepsilon}} \left(e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon}} - 1\right)^{-1} \left(e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon}} + 1\right)^{-1} + \left(13\, e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon} - 1/2\,\varepsilon^{-1}} - 3\right) \\ & e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon} + 1/2\,\varepsilon^{-1}} e^{-1/2\,\frac{(1+\sqrt{1-4\,\varepsilon})t}{\varepsilon}} \left(e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon}} - 1\right)^{-1} \left(e^{1/2\,\frac{\sqrt{1-4\,\varepsilon}}{\varepsilon}} + 1\right)^{-1} \end{aligned} \tag{93}$$

Figure 57 shows the time response of this system.
Figure 57 (d) shows the case where degeneration occurred, resulting in the loss of one boundary condition. Figure 57 (g) is Figure 57 from (a) to (f) represented in one graph.

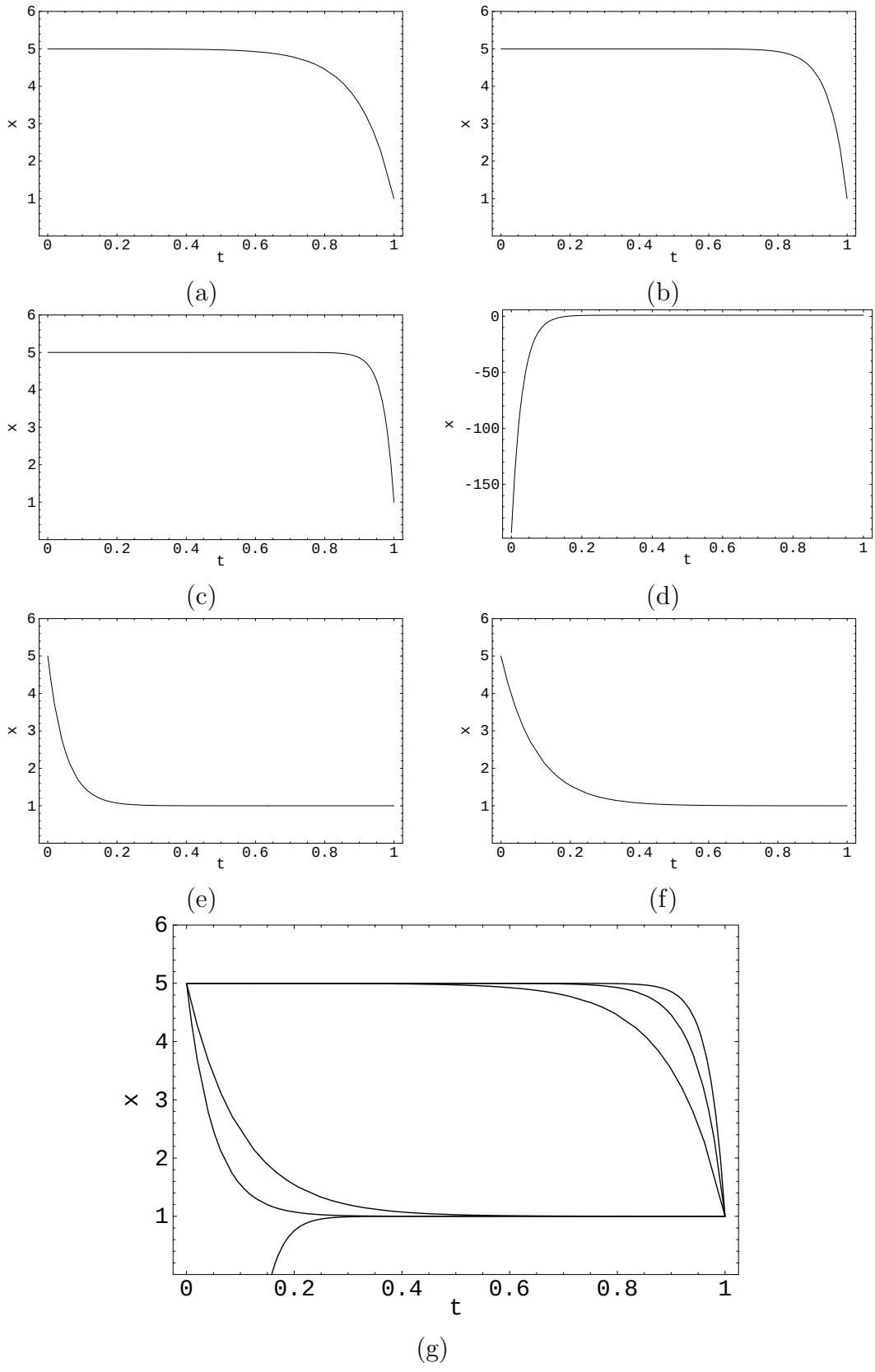


Figure 56: *Boundary Value problem, (a)* $\varepsilon = -0.1$*, (b)* $\varepsilon = -0.05$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.03$*, (e)* $\varepsilon = 0.05$*, (f)* $\varepsilon = 0.1$*, and (g) combined in a single plot*

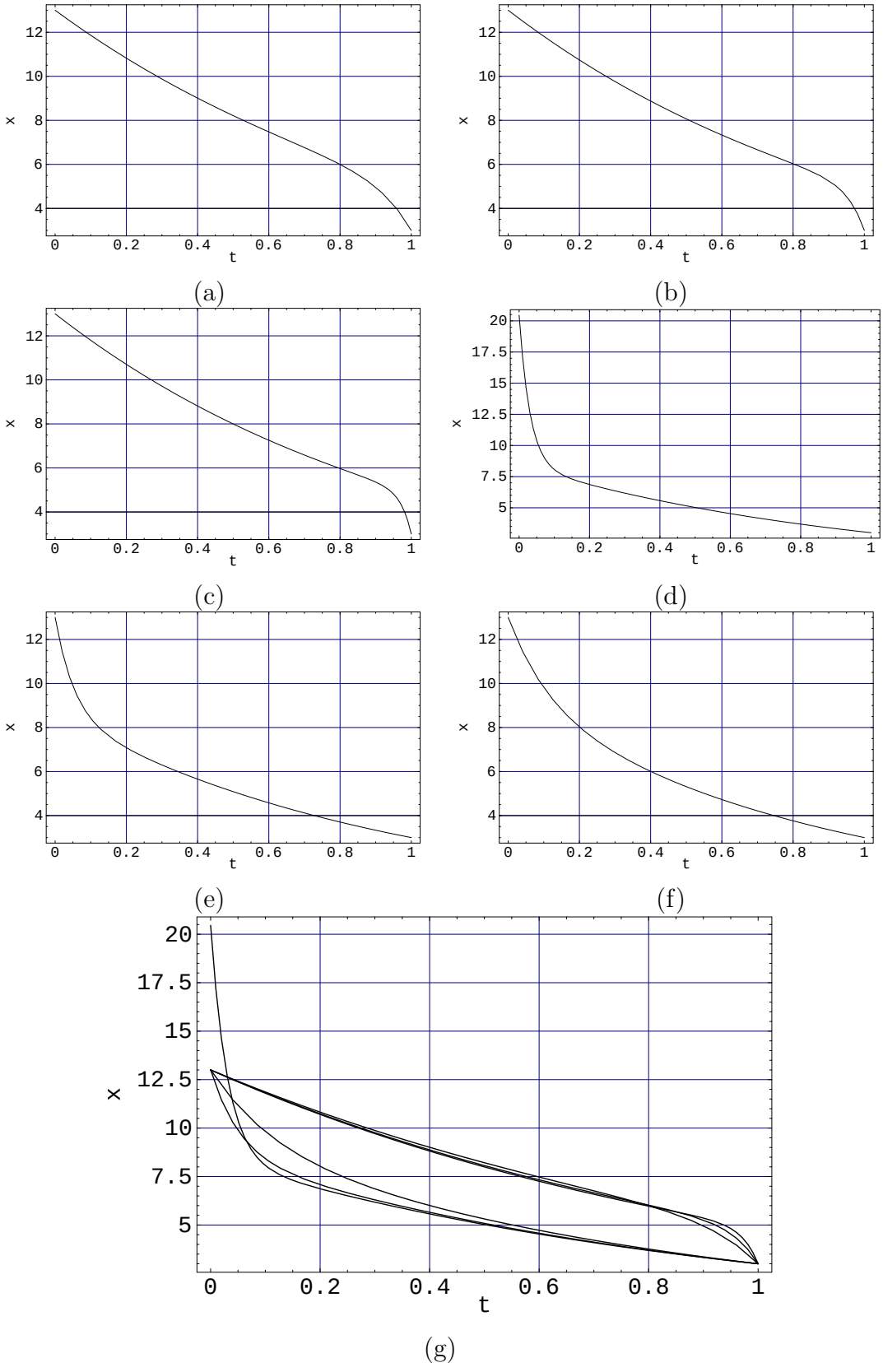


Figure 57: *Boundary Value problem, (a)* $\varepsilon = -0.1$*, (b)* $\varepsilon = -0.05$*, (c)* $\varepsilon = -0.03$*, (d)* $\varepsilon = 0.03$*, (e)* $\varepsilon = 0.05$*, (f)* $\varepsilon = 0.1$*, and (g) combined into a single plot*

ε	$\left(-\infty, -\frac{1}{3}\right]$	$\left(-\frac{1}{3}, 1\right)$	$[1, \infty)$
equilibrium point is a	node	focus	node

Table 5: *Dependence of characteristic of the trajectory upon* ε

Perturbed system

24^{th} **April 1998**
Consider the system

$$\dot{x}_1 = -x_1 + x_2(1 + x_1) + \varepsilon(1 + x_1)^2 \tag{94}$$
$$\dot{x}_2 = -x_1(x_1 + 1) + u. \tag{95}$$

To find the equilibrium point, let $f(x) = 0$ (where $\dot{x} = f(x, u)$, and $u = 0$) and solve the simultaneous equations

$$-x_1 + x_2(1 + x_1) + \varepsilon(1 + x_1)^2 = 0 \tag{96}$$
$$-x_1(x_1 + 1) = 0. \tag{97}$$

The equilibrium point is at $(0, -\varepsilon)$.
The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + 2\varepsilon\,(1 + x_1) + x_2 & 1 + x_1 \\ -1 - 2x_1 & 0 \end{bmatrix}. \tag{98}$$

This Jacobian matrix when evaluated at the equilibrium point gives

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=(0,-\varepsilon)} = \begin{bmatrix} -1 + 3\varepsilon & 1 \\ -1 & 0 \end{bmatrix}. \tag{99}$$

Eigenvalues of A are

$$\lambda_{1,2} = -\frac{1}{2} + \frac{3}{2}\,\varepsilon \pm \frac{1}{2}\,\sqrt{-3 - 6\,\varepsilon + 9\,\varepsilon^2}. \tag{100}$$

The qualitative behaviour of the trajectory is summerized in Table 5 depends on the value of ε, for example

$$-3 - 6\,\varepsilon + 9\,\varepsilon^2 > 0 \tag{101}$$

the trajectory has a node which may either be stable or unstable depends on whether $\lambda_{1,2} < 0$ or > 0 respectively. Moreover when

$$\varepsilon = \frac{1}{3}$$

the trajectory becomes structurally unstable since the eigenvalues on the imaginary axis is vulnerable to perturbations. The equilibrium point becomes a *hyperbolic equilibrium point.*
Figure 58 shows plots versus ε of real and imaginary parts of the eigenvalues.

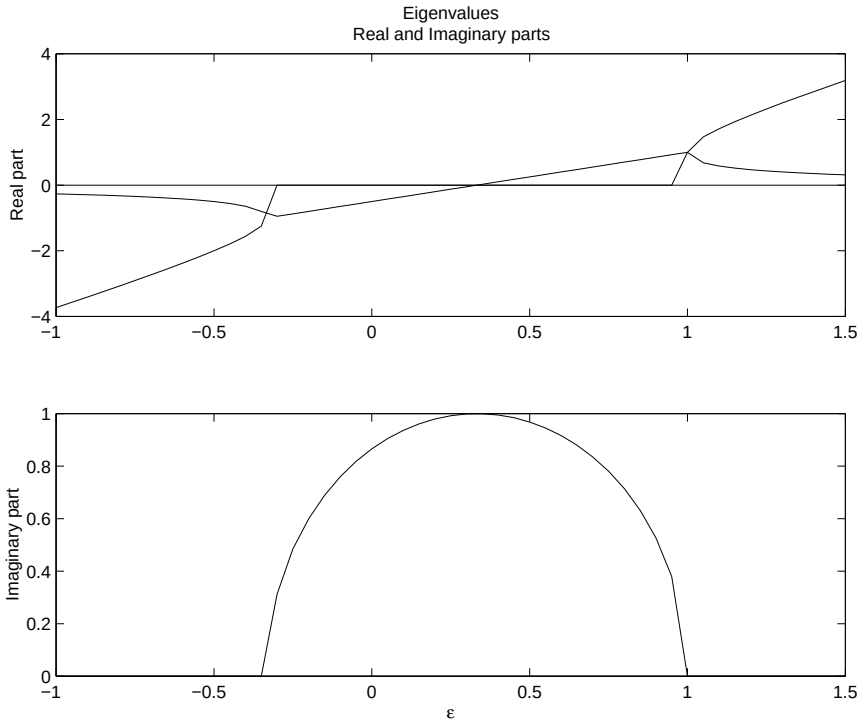


Figure 58: *Eigenvalues, real and imaginary parts*

Domain of attraction

The domain of attraction can be found whenever the matrix A in Equation 99 is a stability matrix (ie all of its eigenvalues are positive definite or $\mathrm{Re}\lambda_i < 0$.) The plot of $\mathrm{Re}\lambda_i$ of the system in Equation 94 and 95 when $u = 0$ is shown in Figure 59 (a) and (b).
As an example, consider a case when $\varepsilon = 0.1$, which means

$$A = \begin{bmatrix} -1 + 3 \times 0.1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -0.7 & 1 \\ -1 & 0 \end{bmatrix}$$

Solving the Lyapunov equation having $Q = I$,

$$PA + A^T P = -Q = -I \tag{102}$$

and get as a result

$$P = \begin{bmatrix} 1.4286 & 0.5 \\ 0.5 & 1.7786 \end{bmatrix}, \quad \Lambda = \mathrm{Re}\lambda_i = \begin{bmatrix} 1.0738 \\ 2.1333 \end{bmatrix}.$$

Then the Lyapunov function can be chosen as

$$V = (1.4286\, x_1 + 0.5 x_2)\, x_1 + (0.5 x_1 + 1.7786\, x_2)\, x_2. \tag{103}$$

And its derivative is

$$\begin{aligned} \dot{V} &= -0.71428\, x_1^3 + x_1^2\,(-3.8572 - 0.6\, x_2) + \\ &\quad x_1\,(-1.51089 + x_2)\,(-0.189107 + x_2) + x_2\,(0.1 + x_2)\,. \end{aligned} \tag{104}$$

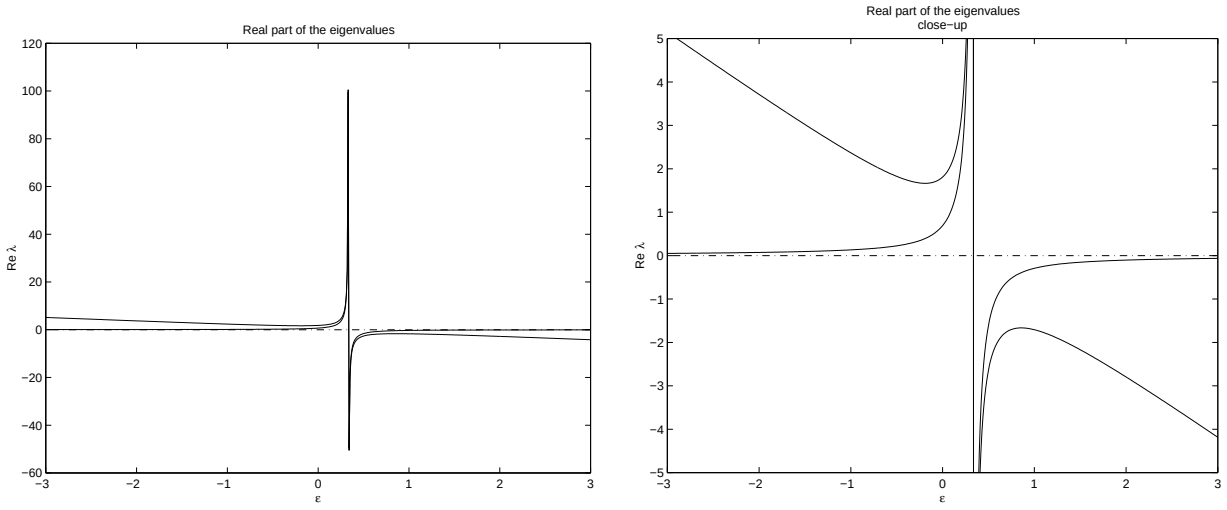


Figure 59: *Real part of eigenvalues (a), and close-up (b)*

$$
\begin{aligned}
&= 0.28572\,x_1 - 3.8572\,x_1^2 - 0.71428\,x_1^3 + 0.1\,x_2 - 1.7\,x_1x_2 - 0.6\,x_1^2x_2 + x_2^2 + x_1x_2^2 \quad (105)\\
&\leq -3.8572\|x\|_2 + 0.29\|x\|_2 + 0.72\|x\|_2^3 + 0.1\|x\|_2 + \sqrt{1.72^2+1}\|x\|_2^2 + \\
&\quad \left(\frac{1}{2}\|x\|_2^2\right)\left(\sqrt{0.6^2+1}\|x\|_2\right) \quad (106)\\
&= -1.4949\|x\|_2 + 1.8862\|x\|_2^3 \quad < 0, \quad \text{for } \|x\|_2^2 = r^2 < 0.7925. \quad (107)
\end{aligned}
$$

Thus $\dot{V}(x)$ is negative definite in a ball ([Kha96],Chapter 4)

$$\mathcal{D} = \left\{x \in \mathcal{R}^2 \,|\, \|x\|_2 < r\right\} \tag{108}$$

of radius

$$r = \sqrt{0.7925} = 0.8903.$$

The simplest estimate is the set

$$\Omega_c = \left\{x \in \mathcal{R}^2 \,|\, V(x) \leq c\right\}. \tag{109}$$

Then

$$c < \min_{\|x\|_2 = r} V(x) = \lambda_{min}(P) r^2 \tag{110}$$

leads to

$$c = 0.8 \quad < \quad 1.0738 \times 0.7925 = 0.8510.$$

Then better estimates of the region of attraction can be obtained by finding the largest ball on which $\dot{V}(x)$ is negative definite. This can be done via the *trajectory reversing method* ([Kha96], Chapter 4.2.) This method is essentially done by backward integrating the state equations using points which lie on the surface of the boundary of the estimate of the region of attraction. Backward integration of the state equations can be done by forward integrating the new state equations

$$\dot{x} = -f(x). \tag{111}$$

Let $u = 0$ and write Equation 94 and 95 in the form of Equation 111

$$\dot{x}_1 = x_1 - x_2(1 + x_1) - \varepsilon(1 + x_1)^2 \tag{112}$$
$$\dot{x}_2 = x_1(x_1 + 1). \tag{113}$$

Then simulate these new equations using the 20 obtained by solving

$$\theta_k = \frac{2\pi k}{20}, \quad \text{and} \tag{114}$$
$$\rho_k^2 \begin{bmatrix} \cos\theta_k \\ \sin\theta_k \end{bmatrix}^T P \begin{bmatrix} \cos\theta_k \\ \sin\theta_k \end{bmatrix} = 0.89 \tag{115}$$

as initial conditions.
The contours in Figure 60 are the estimated region of attraction obtained by the method of trajectory-reversing described above. Backward integration was done over the interval $[0, t_i]$, $i = 1, 2, 3, \ldots, 44$. At $t_i \approx 0.4$ seconds the simulation encountered singularities and the further simulation is no longer possible. The wedge-like shape was due to backward integration of three points on W_0 which approach infinity rapidly.

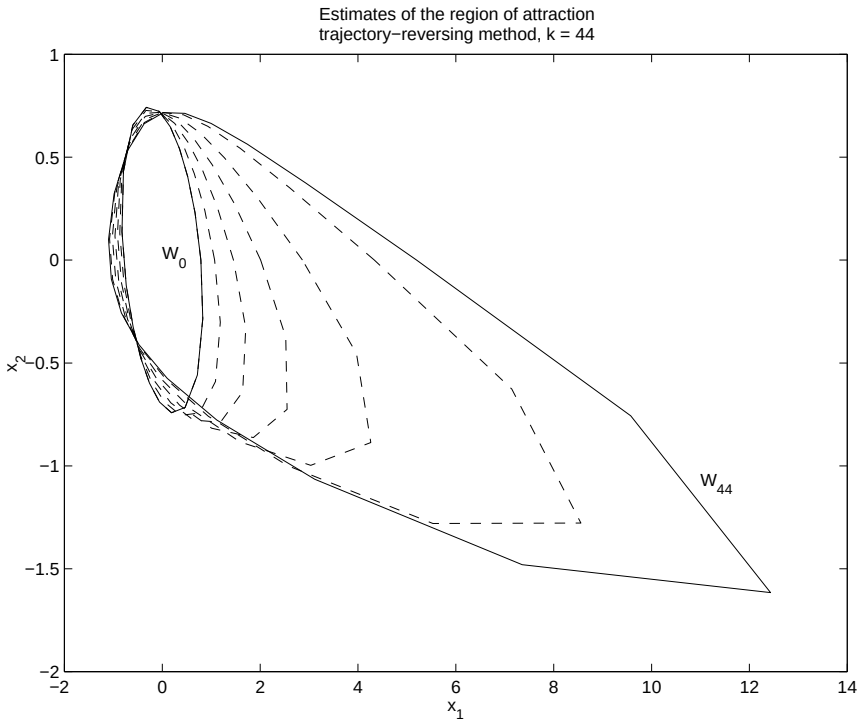


Figure 60: *Estimates of the region of attraction ($\varepsilon = -1$)*

Sensitivity analysis of the system

The procedure used to do the stability analysis is referred to a reference [Kha96]. Let the nominal value of perturbing parameter be $\varepsilon = 1$. The nominal system is then given by $(u = 0)$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2(1 + x_1) + (1 + x_1)^2 && (116) \\ \dot{x}_2 &= -x_1(x_1 + 1). && (117) \end{aligned}$$

The Jacobian matrix $\frac{\partial f}{\partial x}$ was given by Equation 98 and $\frac{\partial f}{\partial \varepsilon}$ is

$$\frac{\partial f}{\partial \varepsilon} = \begin{bmatrix} (1 + x_1)^2 \\ 0 \end{bmatrix}. \tag{118}$$

These Jacobian matrice when evaluated at the nominal value of ε give

$$\left.\frac{\partial f}{\partial x}\right|_{nominal} = \begin{bmatrix} -1 + 2(1 + x_1) + x_2 & 1 + x_1 \\ -1 - 2x_1 & 0 \end{bmatrix}, \text{ and } \left.\frac{\partial f}{\partial \varepsilon}\right|_{nominal} = \begin{bmatrix} (1 + x_1)^2 \\ 0 \end{bmatrix}. \tag{119}$$

Augmented equations are ([Kha96], Equation 2.11)

$$\begin{aligned} \dot{x} &= f(t, x, \lambda), & x(t_0) &= x_0 \\ \dot{x}_\lambda &= \left[\tfrac{\partial f(t,x,\lambda)}{\partial x}\right] x_\lambda + \left[\tfrac{\partial f(t,x,\lambda)}{\partial \lambda}\right], & x_\lambda(t_0) &= 0 \end{aligned} \tag{120}$$

Let

$$S = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \varepsilon} \\ \frac{\partial x_2}{\partial \varepsilon} \end{bmatrix} \tag{121}$$

Then the four augmented equations (two original equations and two variational equations) are

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2(1 + x_1) + (1 + x_1)^2, & x_1(0) &= x_{10} \\ \dot{x}_2 &= -x_1(x_1 + 1), & x_2(0) &= x_{20} \\ \dot{x}_3 &= (-1 + 2(1 + x_1) + x_2)\, x_3 + (1 + x_1)\, x_4 + (1 + x_1)^2, & x_3(0) &= 0 \\ \dot{x}_4 &= (-1 - 2x_1)\, x_3, & x_4(0) &= 0. \end{aligned} \tag{122}$$

Figure 61 shows the plot of sensitivity functions. It shows that x_2 is more sensitive to ε than x_1.

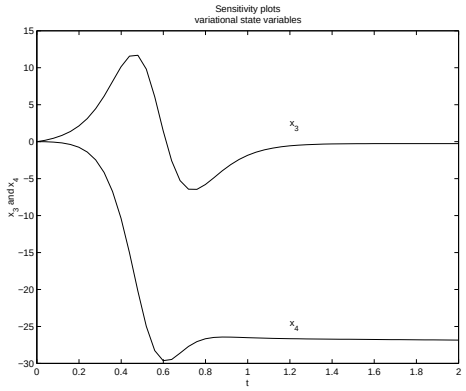


Figure 61: *Sensitivity plot*

Response with various values of ε with no input

Phase planes of this system ($u = 0$) are shown in Figure 62 ($\varepsilon = -1.5$), Figure 63 ($\varepsilon = -1$), Figure 64 ($\varepsilon = -0.1$), Figure 65 ($\varepsilon = 0$), Figure 66 ($\varepsilon = 0.3$), Figure 67 ($\varepsilon = 1.3$) and Figure 68 ($\varepsilon = 1.7$).

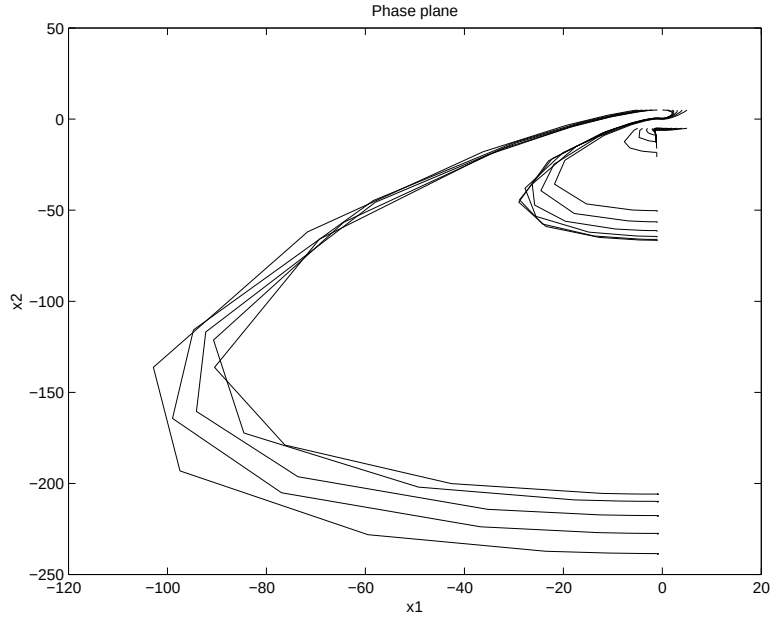


Figure 62: *Phase plane.* $\varepsilon = -1.5$

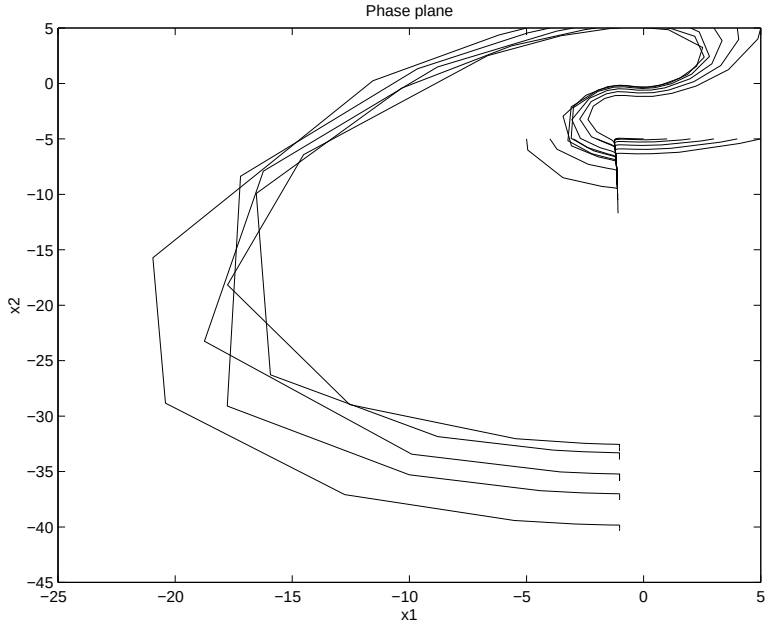


Figure 63: *Phase plane.* $\varepsilon = -1$

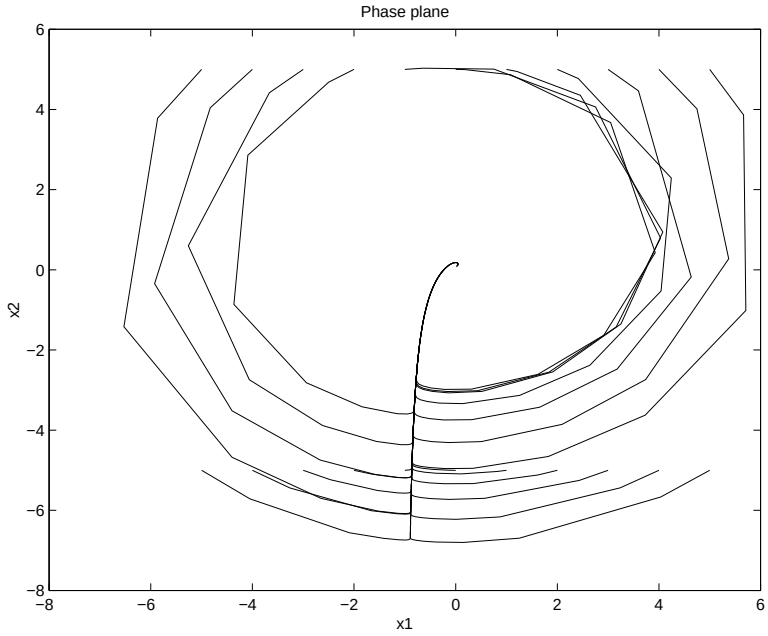


Figure 64: *Phase plane.* $\varepsilon = -0.1$

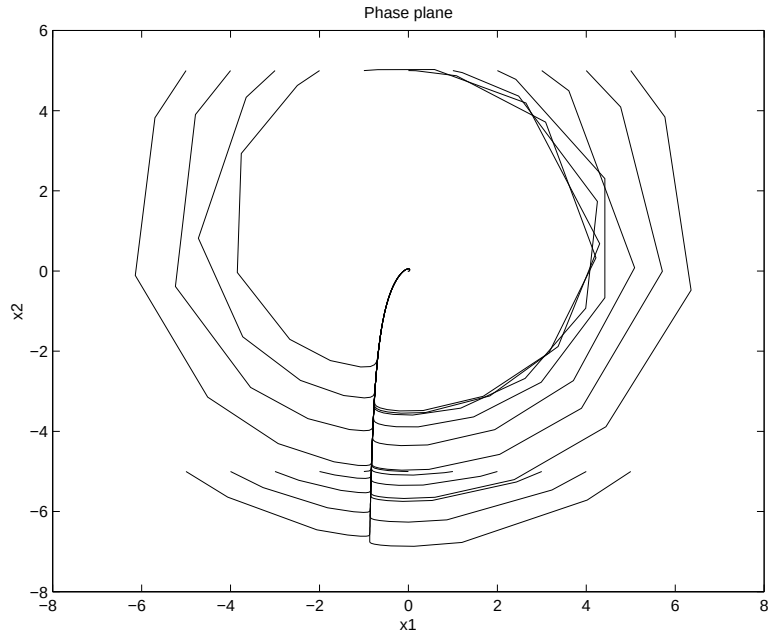


Figure 65: *Phase plane.* $\varepsilon = 0$

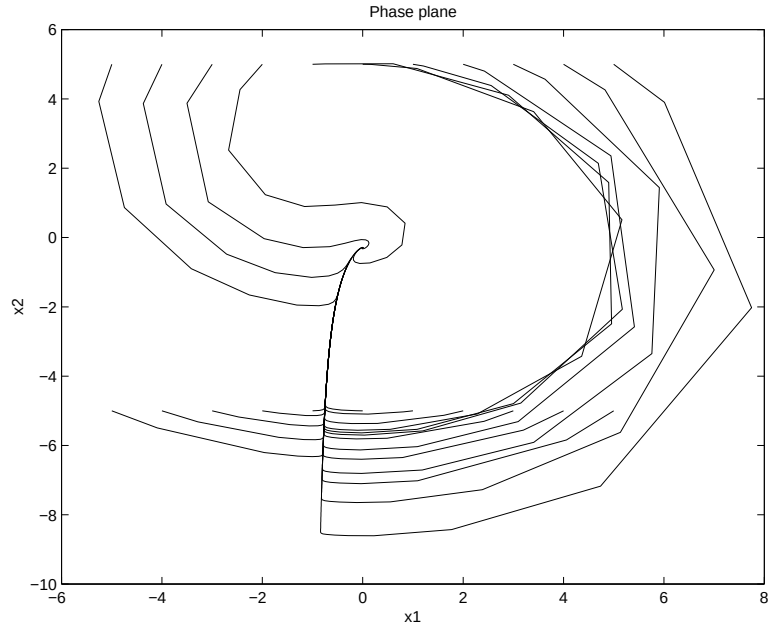


Figure 66: *Phase plane.* $\varepsilon = 0.3$

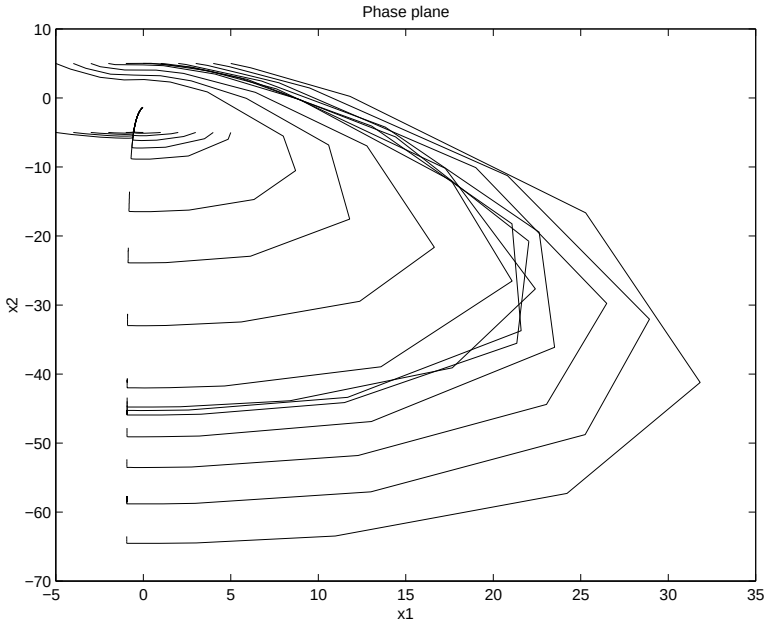


Figure 67: *Phase plane.* $\varepsilon = 1.3$

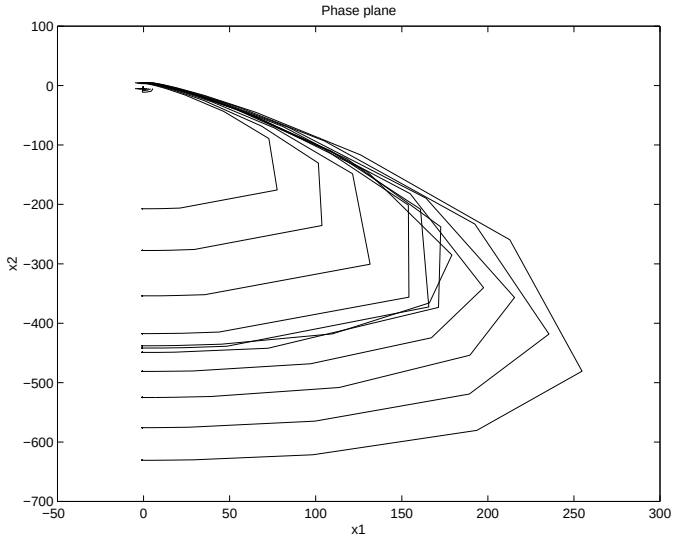


Figure 68: *Phase plane.* $\varepsilon = 1.7$

With a sign function in feedback

Introducing $u = \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$ as an input and obtain Figure 69 ($\varepsilon = -1.5$), Figure 70 ($\varepsilon = -1$), Figure 71 ($\varepsilon = -0.1$), Figure 72 ($\varepsilon = 0$), Figure 73 ($\varepsilon = 0.3$), Figure 74 ($\varepsilon = 1.3$) and Figure 75 ($\varepsilon = 1.7$)

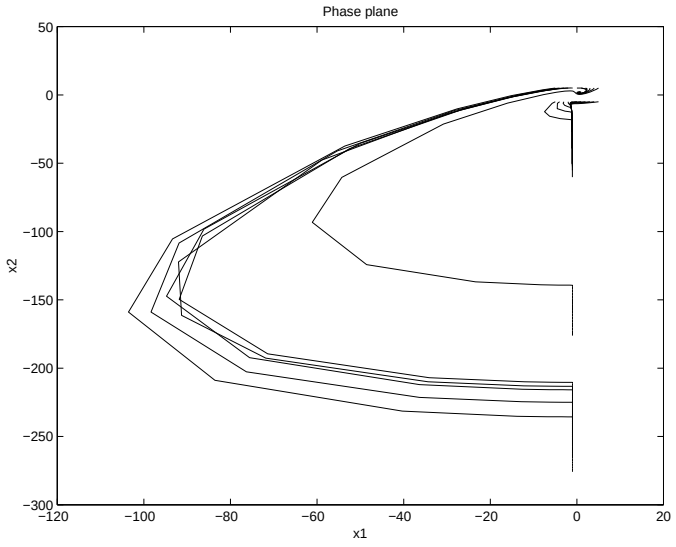


Figure 69: *Phase plane.* $\varepsilon = -1.5$ *with VS input.*

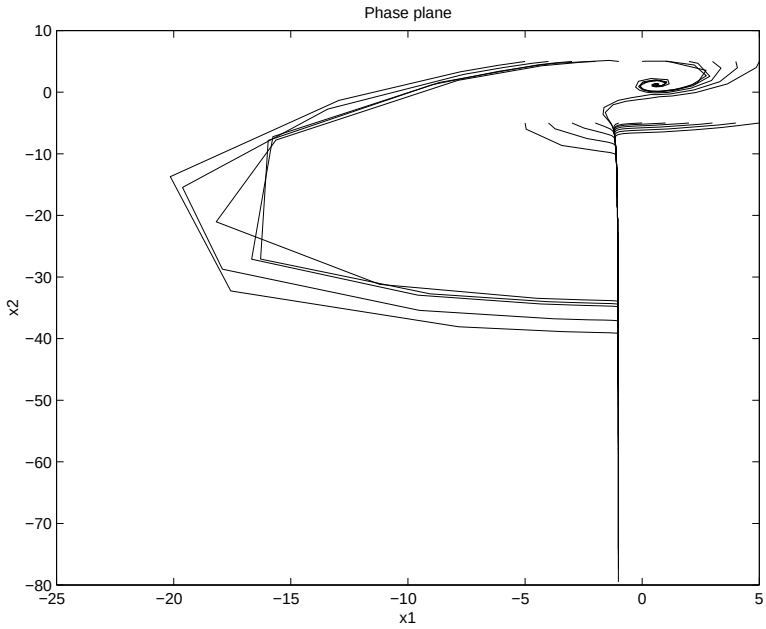


Figure 70: *Phase plane.* $\varepsilon = -1$ *with VS input.*

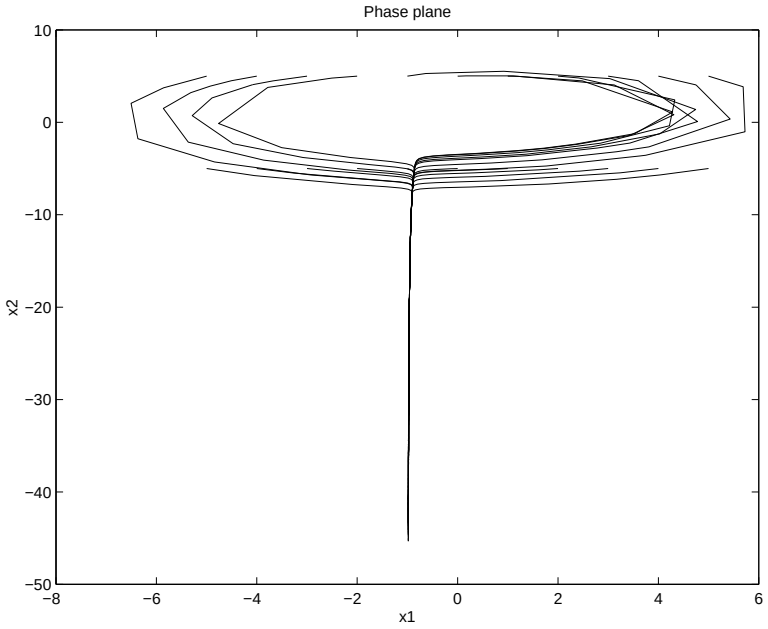


Figure 71: *Phase plane.* $\varepsilon = -0.1$ *with VS input.*

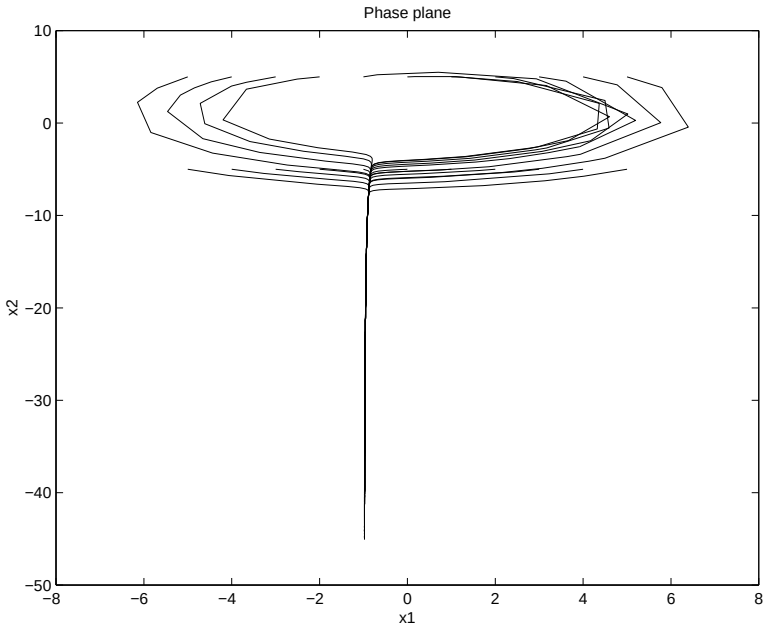


Figure 72: *Phase plane.* $\varepsilon = 0$ *with VS input.*

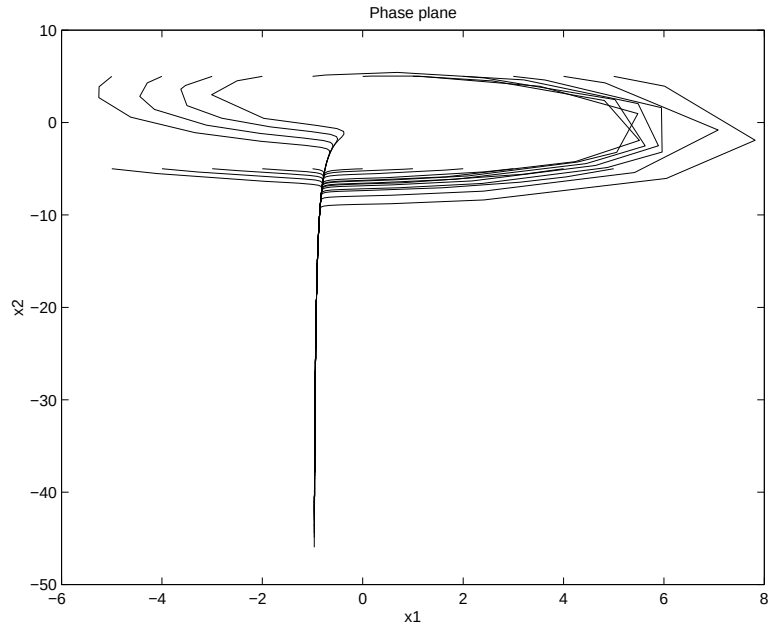


Figure 73: *Phase plane.* $\varepsilon = 0.3$ *with VS input.*

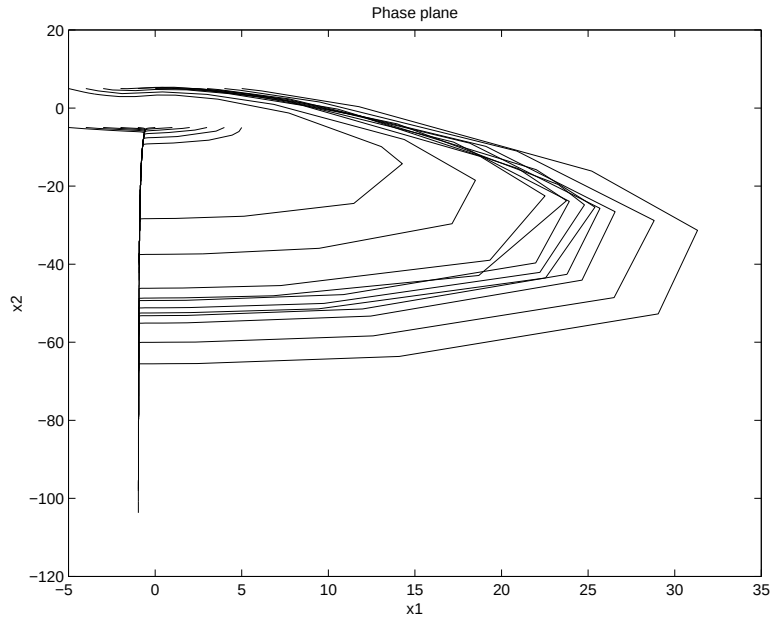


Figure 74: *Phase plane.* $\varepsilon = 1.3$ *with VS input.*

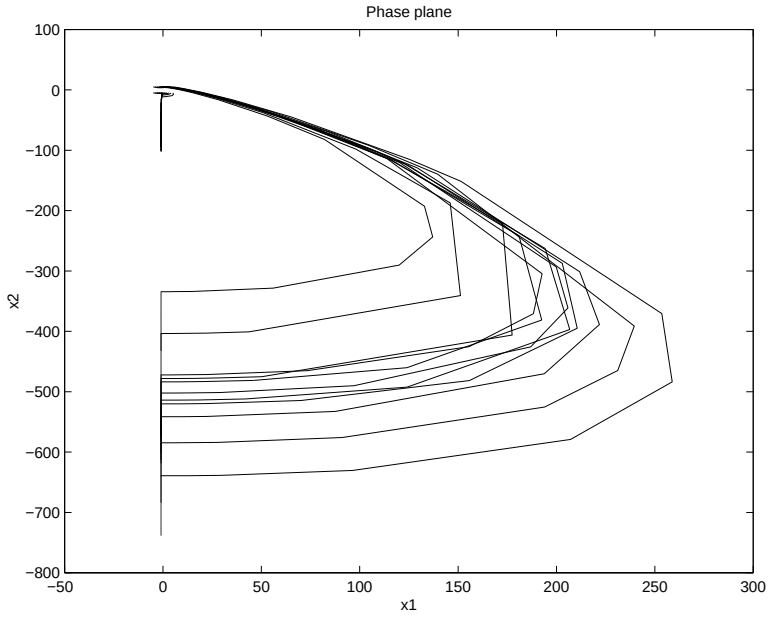


Figure 75: *Phase plane.* $\varepsilon = 1.7$ *with VS input.*

Variable structure control

Let the input be

$$u = -K\text{sgn}\ (cx_1 + x_2)\,, \tag{123}$$

where $K, c \in \mathcal{R}$ and the switching hyperplane is $s = cx_1 + x_2$.
And let the Lyapunov function be

$$V = \frac{1}{2}s^2.$$

Then

$$\dot{s} = -\left(1 + (1 + x_1)\right) + c\left(-x_1 + \varepsilon\left(1 + x_1^2\right) + (1 + x_1)\,x_2\right) - K\text{sgn}\ (cx_1 + x_2) \tag{124}$$

and

$$\begin{aligned} \dot{V} &= s\dot{s} \\ &= (cx_1 + x_2)\left(-\left(1 + (1 + x_1)\right) + c\left(-x_1 + \varepsilon\left(1 + x_1^2\right) + (1 + x_1)\,x_2\right) - \right. \\ &\quad \left. K\text{sgn}\ (cx_1 + x_2)\right) \end{aligned} \tag{125}$$

One could plot Equation 125 for various values of ε (as has been done in dmy earlier works,) in order to find out whether this system is stable. But at this moment let us look at another possiblity of analysing a variable structure control.

Variable structure control with a hyperbolic tangent function

This section introduces and investigates how a hyperbolic tangent function could be used to estimate the sign function. One great advantage of the hyperbolic tangent function over the sign function is that it can be differentiated, as

$$\frac{d\tanh(x)}{dx} = \text{sech}(x).$$

Figure 76 shows a juxtaposition of the $\tanh(\gamma x)$ and $\text{sgn}\,(x)$, $\gamma = 1, 5, 10, 50$, and 100. The figure shows that the hyperbolic tangent function can be used to approximate the sign function very well.

Let the input to the system is

$$u = K\tanh\left(\gamma\,(cx_1 + x_2)\right). \tag{126}$$

Equation 126 is an approximation of Equation 123.
Equation 94 and 95 with Equation 126 as an input would turn into

$$\dot{x}_1 = -x_1 + x_2(1 + x_1) + \varepsilon(1 + x_1)^2 \tag{127}$$

$$\dot{x}_2 = -x_1(x_1 + 1) - K\tanh\left(\gamma\,(cx_1 + x_2)\right), \tag{128}$$

where γ, c, K are constants.
The Jacobian matrix is then

$$A = \frac{\partial f(x)}{\partial x} = \begin{bmatrix} -1 + 2\varepsilon\,(1 + x_1) + x_2 & 1 + x_1 \\ -1 - 2x_1 - c\gamma K\text{sech}^2\left(\gamma\,(cx_1 + x_2)\right) & -\gamma K\text{sech}^2\left(\gamma\,(cx_1 + x_2)\right) \end{bmatrix}. \tag{129}$$

If the location of the equilbrium point is known and one would like to find a domain of attraction, one could use the following procedure

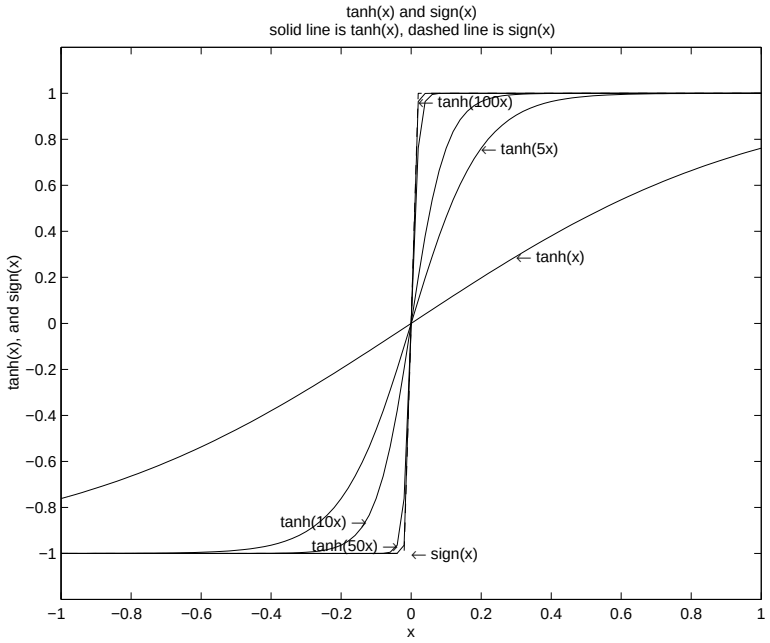


Figure 76: *Hyperbolic tangent function compared with sign function*

1. evaluate the Jacobian matrix at the equilibrium point to find find the matrix A of the system linearized around the equilibrium point,

2. solve the Lyapunov equation for P,

3. obtain the Lyapunov function from
$$V = x^T P x, \tag{130}$$

4. find the first estimate of the domain of attraction by looking at terms in the Lyapunov function, and

5. plot better estimates of the region of attraction by the trajectory-reversing method.

But in our case instead of solving the Equation 127 and 128 which could prove to be quite difficult, I will simplify the problem by letting

$$\varepsilon = 0.1, \quad K = 30, \quad \gamma = 50, \quad \text{and} \quad c = 1.3,$$

and plot the response in state space as Figure 77. From Figure 77 the response is stable.

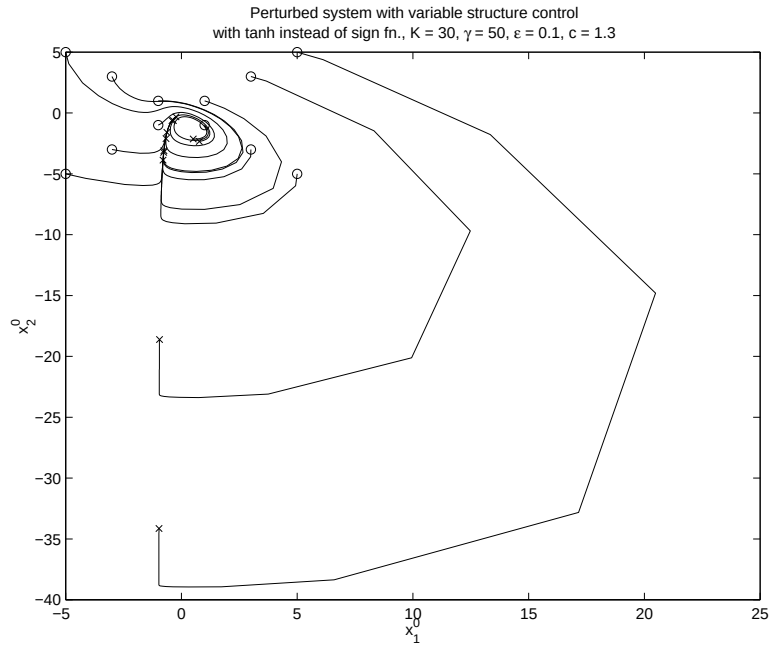


Figure 77: *Hyperbolic tangent variable structure control, state plane*

Since in Figure 77 the equilibrium point is in the vicinity of the origin it is possible to locate it by calculating $f_1(x)$ and $f_2(x)$ and look for a point where these two functions are at minimum value. The calculation done using Algorithm 1 shows that the equilibrium point is at the origin and

$$\begin{array}{rcll} f_1|_{equilibrim} & = & -x_1 + x_2(1+x_1) + \varepsilon(1+x_1)^2 & = \quad \varepsilon \\ f_2|_{equilibrim} & = & -x_1(x_1+1) - K\tanh\left(\gamma\left(cx_1 + x_2\right)\right) & = \quad 0. \end{array} \tag{131}$$

Algorithm 1 `f1_eq = 10; f2_eq = 10; x1_eq = 0, x2_eq = 0;`

```
for x1 = -5 upto 5 step 0.1
  for x2 = -5 upto 5 step 0.1
    calculate f1(x) and f2(x);
    if (|f1(x)| < f1_eq) and (|f2(x)| < f2_eq)
      x1_eq = x1; x2_eq = x2; f1_eq = f1(x); f2_eq = f2(x);
    endif
  endfor
endfor
```

Do Algorithm 1 again for various values of ε then plot Figure 78, which shows that

$$(x_1, x_2)|_{equilibrium} = (0, 0) \quad \text{for} \quad \varepsilon < 0.27 \quad (K = 30, \gamma = 50, \text{and } c = 1.3).$$

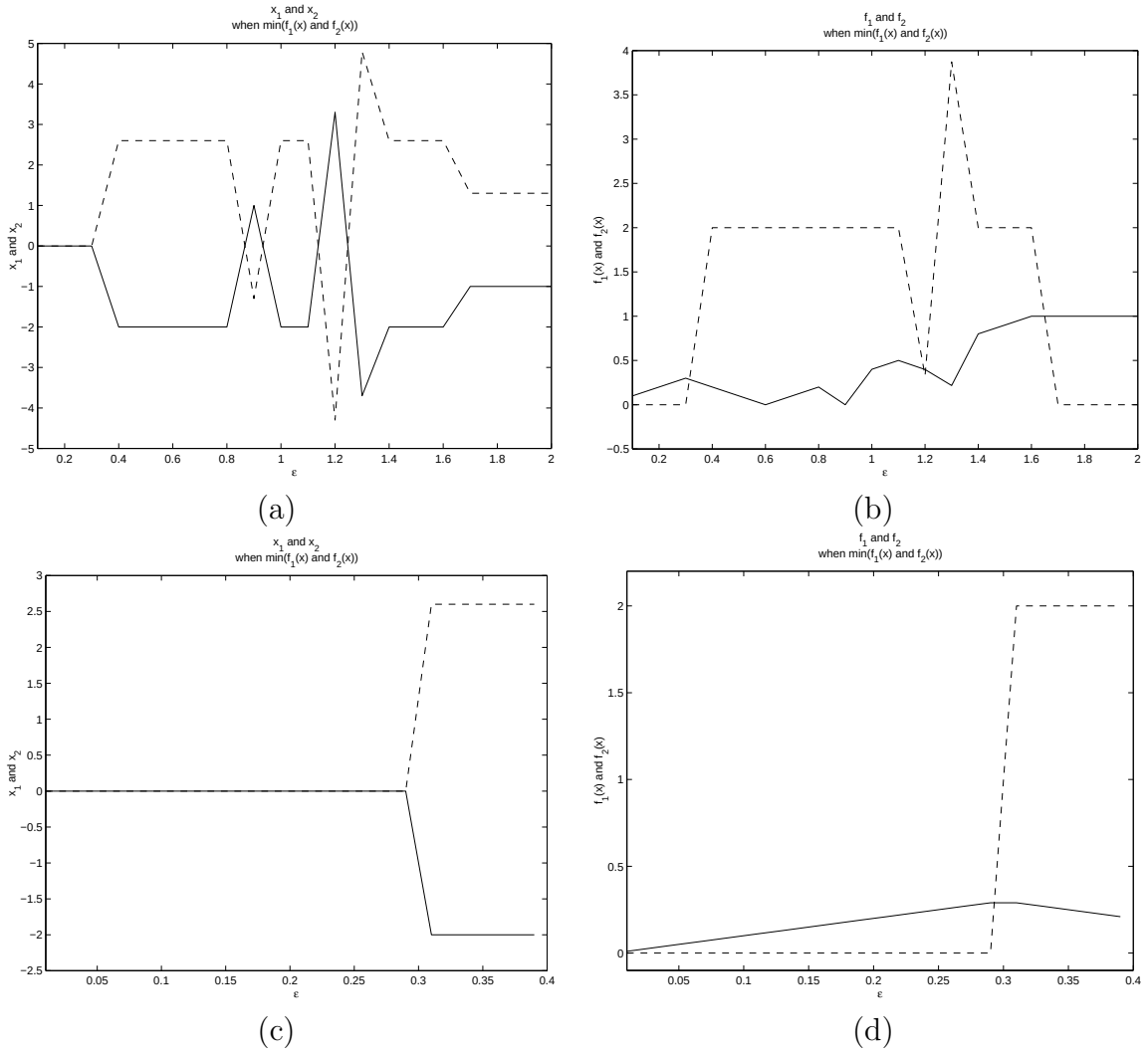


Figure 78: *(a), (c),* x_1 *and* x_2*; (b), (d)* $f_1(x)$ *and* $f_2(x)$ *when* $\min(f_1(x))$ and $\min(f_2(x))$

The Jacobian matrix at the equilibrium point for $\varepsilon = 0.1 < 0.27$ is then

$$A = \begin{bmatrix} -1 & 1 \\ -1951 & -1500 \end{bmatrix}$$

and the eigenvalues are correspondingly

$$\lambda_1 = -2.3, \quad \text{and} \quad \lambda_2 = -1498.7.$$

Then solve the Lyapunov equation $AP + PA^T = -I$ to get P which is

$$P = \begin{bmatrix} 0.2175 & -0.2825 \\ -0.2825 & 0.3677 \end{bmatrix},$$

and is a positive definite matrix since the eigenvalues are all positive (0.003 and 0.5849.)

Let the Lyapunov function be

$$V = x^T P x = (0.2175\,x_1 - 0.2825\,x_2)\,x_1 + (-0.2825\,x_1 + 0.3677\,x_2)\,x_2.$$

Then the derivative of Lyapunov function is

$$\begin{aligned}
\dot{V} &= (0.2175\,x_1 - 0.2825\,x_2)\,\dot{x}_1 + x_1\,(0.2175\,\dot{x}_1 - 0.2825\,\dot{x}_2) + x_2\,(-0.2825\,\dot{x}_1 + 0.3677\,\dot{x}_2) + \\
&\quad (-0.2825\,x_1 + 0.3677\,x_2)\,\dot{x}_2 \qquad (132)\\
&= (0.2175\,x_1 - 0.2825\,x_2)\left(-x_1 + 0.1\,(1+x_1)^2 + (x_1)\,x_2\right) + \\
&\quad x_1\left(0.2175\left(-x_1 + 0.1\,(1+x_1)^2 + (1+x_1)\,x_2\right) - 0.2825\,(-(x_1\,(1+x_1)) - \right.\\
&\quad \left. 30\tanh(50\,(1.3\,x_1 + x_2)))\right) + x_2\left(-0.2825\left(-x_1 + 0.1\,(1+x_1)^2\right) + (1+x_1)\,x_2\right) + \\
&\quad 0.3677\,(-(x_1\,(1+x_1)) - 30\tanh(50\,(1.3\,x_1 + x_2))) + \\
&\quad (-0.2825\,x_1 + 0.3677\,x_2)\,(-(x_1\,(1+x_1)) - 30\tanh(50\,(1.3x_1 + x_2))) \qquad (133)\\
&= 0.6085\,x_1^3 + x_1^2\,(0.217 - 0.3569\,x_2) + (-0.0565 - 0.565\,x_2)\,x_2 - \\
&\quad 0.565\,x_1\,(-0.442364 + x_2)\,(0.174045 + x_2) + (16.95\,x_1 - \\
&\quad 22.062\,x_2)\tanh(65\,x_1 + 50x_2) \qquad (134)\\
&= 0.0435\,x_1 + 0.217\,x_1^2 + 0.6085\,x_1^3 - 0.0565\,x_2 + 0.1516\,x_1x_2 - 0.3569\,x_1^2x_2 - \\
&\quad 0.565\,x_2^2 - 0.565\,x_1x_2^2 + 16.95\,x_1\tanh(50\,(1.3\,x_1 + x_2)) - \\
&\quad 22.062\,x_2\tanh(50\,(1.3\,x_1 + x_2)) \qquad (135)
\end{aligned}$$

Equation 134 is in a simplified form and Equation 135 in an expanded form.
In polar coordinates

$$\begin{aligned}
x_1 &= \rho\sin\theta, \quad \text{and} \qquad (136)\\
x_2 &= \rho\cos\theta. \qquad (137)
\end{aligned}$$

The Lyapunov function in Equation was derived from state equations which are written in $x_1 - x_2$ Cartesian coordinates. Change of the variables x_1 and x_2 in Equation into a $\rho - \theta$ polar-coordinates results in

$$\begin{aligned}
V(\rho,\theta) &= \rho\cos(\theta)\,(0.3677\,\rho\cos(\theta) - 0.2825\rho\sin(\theta)) + \rho\sin(\theta)\,(-0.2825\,\rho\cos(\theta) + \\
&\quad 0.2175\,\rho\sin(\theta)) \qquad (138)\\
&= \rho^2\left(0.3677\cos^2(\theta) - 0.565\cos(\theta)\sin(\theta) + 0.2175\sin^2(\theta)\right). \qquad (139)
\end{aligned}$$

Similarly, the derivative of the Lyapunov function in Equation 132 to 135 becomes

$$\begin{aligned}
\dot{V}(\rho,\theta) &= (-0.2825\,\rho\cos(\theta) + 0.2175\,\rho\sin(\theta))\,(-\rho\sin(\theta) + \rho\cos(\theta)\,(1 + \rho\sin(\theta)) + \\
&\quad 0.1\,(1+\rho\sin(\theta))^2) + \rho\sin(\theta)\,(0.2175\,(-\rho\sin(\theta) + \rho\cos(\theta)\,(1+\rho\sin(\theta)) + \\
&\quad 0.1\,(1+\rho\sin(\theta))^2) - 0.2825\,(-\rho\sin(\theta)\,(1+\rho\sin(\theta)) - 30\tanh(50\,(\rho\cos(\theta) + \\
&\quad 1.3\,\rho\sin(\theta))))) + \rho\cos(\theta)\,(-0.2825\,(-\rho\sin(\theta) + \rho\cos(\theta)\,(1+\rho\sin(\theta)) + \\
&\quad 0.1\,(1+\rho\sin(\theta))^2) + 0.3677\,(-\rho\sin(\theta\,(1+\rho\sin(\theta)) - 30\tanh(50\,(\rho\cos(\theta) + \\
&\quad 1.3\,\rho\sin(\theta))))) + (0.3677\,\rho\cos(\theta) - 0.2825\,\rho\sin(\theta))\,(-\rho\sin(\theta)\,(1+\rho\sin(\theta)) - \\
&\quad 30\tanh(50\,(\rho\cos(\theta) + 1.3\,\rho\sin(\theta)))) \qquad (140)\\
&= \rho\left(-0.565\,\rho\cos^2(\theta)\,(1+\rho\sin(\theta)) + \cos(\theta)\,(-0.0565 + 0.1516\,\rho\sin(\theta) - \right.\\
&\quad \left. 0.3569\,\rho^2\sin^2(\theta)\right) + \sin(\theta)\left(0.0435 + 0.217\,\rho\sin(\theta) + 0.6085\,\rho^2\sin^2(\theta)\right) + \\
&\quad (-22.062\cos(\theta) + 16.95\sin(\theta))\tanh(50\,\rho\cos(\theta) + 65\,\rho\sin(\theta))) \qquad (141)
\end{aligned}$$

Notice that both the Lyapunov function and its derivative was derived in the Cartesian coordinates x_1, x_2 domain. The change of variable was only adopted to facilitate the consideration of the equations. This is different from deriving a new Lyapunov function from the polar-coordinates ρ, θ domain whose states are

$$r = \sqrt{x_1^2 + x_2^2} \tag{142}$$
$$\theta = \tan^{-1}\frac{x_2}{x_1}. \tag{143}$$

Following is shown a wrong method in deriving the state equations in polar-coordinate, then the right method is given afterward.

The wrong method is to differentiate Equation 142, 143 and then substitute x, $\dot{x}$ in the right hand side of the equation obtained. To illustrate that this method does not work, try solving for state equations with parameters as before.

$$\begin{aligned}\dot{r} &= \frac{d}{dt}\sqrt{x_1^2 + x_2^2} \\ &= \frac{2x_1\dot{x}_1 + 2x_2\dot{x}_2}{2\sqrt{x_1^2 + x_2^2}}\end{aligned} \tag{144}$$

$$\begin{aligned}\dot{\theta} &= \frac{d}{dt}\tan^{-1}\frac{x_2}{x_1} \\ &= \frac{-\frac{x_2\dot{x}_1}{x_1^2} + \frac{\dot{x}_2}{x_1}}{1 + \frac{x_2^2}{x_1^2}}.\end{aligned} \tag{145}$$

Substitute Equation 136, 137 as well as

$$\dot{x}_1 = \sin\theta\,\dot{\rho} + \cos\theta\,\rho\dot{\theta} \tag{146}$$
$$\dot{x}_2 = \cos\theta\,\dot{\rho} - \rho\,\sin\theta\,\dot{\theta} \tag{147}$$

into Equation 144 and 145 to obtain

$$\begin{aligned}\dot{r} &= \frac{2\rho\,\sin(\theta)(\sin(\theta)\dot{\rho} + \cos(\theta)\rho\dot{\theta}) + 2\cos(\theta)\rho(\cos(\theta)\dot{\rho} - \rho\,\sin(\theta)\dot{\theta})}{2\sqrt{\cos^2(\theta)\rho^2 + \rho^2\sin^2(\theta)}} \\ &= \dot{r} \\ \dot{\theta} &= \frac{-\cot(\theta)\csc(\theta)(\sin(\theta)\dot{\rho} + \cos(\theta)\rho\dot{\theta}) + \csc(\theta)(\cos(\theta)\dot{\rho} - \rho\,\sin(\theta)\dot{\theta})}{\rho\,(1 + \cot^2(\theta))} \\ &= -\dot{\theta}.\end{aligned}$$

Which is obviously wrong!

† † †

The correct method is as follows.
From Equation 127 and 128 substitute Equation 136, 137, 146, and 147 to obtain two simultaneous equations

$$\sin(\theta)\dot{\rho} + \cos(\theta)\rho\dot{\theta} = -(\rho\sin(\theta)) + \cos(\theta)\rho(1 + \rho\,\sin(\theta)) + \varepsilon\,(1 + \rho\,\sin(\theta))^2 \tag{148}$$
$$\cos(\theta)\dot{\rho} - \rho\,\sin(\theta)\dot{\theta} = -\rho\,\sin(\theta)(1 + \rho\,\sin(\theta)) - K\tanh(\gamma(\cos(\theta)\rho + c\rho\,\sin(\theta))) \tag{149}$$

which are then solved for $\dot{\rho}$ and $\dot{\theta}$.
The state equations are then

$$\begin{aligned}
\dot{\rho} &= -\frac{1}{\cos^2(\theta)+\sin^2(\theta)}\left(-\varepsilon\,\sin(\theta)+\rho\,\sin^2(\theta)-2\varepsilon\rho\,\sin^2(\theta)-\varepsilon\rho^2\sin^3(\theta)+\right.\\
&\quad \left. K\,\cos(\theta)\tanh\left(\gamma(\rho\,\cos(\theta)+c\rho\,\sin(\theta))\right)\right) && (150)\\
&= \sin(\theta)(\varepsilon+(-1+2\varepsilon)\rho\,\sin(\theta)+\varepsilon\rho\,\sin(\theta))-K\,\cos(\theta)\tanh(\gamma\rho(\cos(\theta)+c\,\sin(\theta))) && (151)\\
\dot{\theta} &= -\frac{1}{\rho\,\cos^2(\theta)+\rho\,\sin^2(\theta)}\left(\cos(\theta)\left(\rho\,\sin(\theta)-\rho\,\cos(\theta)\left(1+\rho\,\sin(\theta)-\varepsilon\left(1+\rho\,\sin(\theta)\right)^2\right)-\right.\right.\\
&\quad \left.\left. \sin(\theta)\left(\rho\,\sin(\theta)(1+\rho\,\sin(\theta))+K\,\tanh\left(\gamma(\rho\,\cos(\theta)+c\rho\,\sin(\theta))\right)\right)\right)\right) && (152)\\
&= \frac{1}{4\rho}\left(\varepsilon\left(4+\rho^2\right)\cos(\theta)+\rho\left(4-\varepsilon\rho\,\cos(3\theta)+4\rho\,\sin(\theta)-2\,\sin(2\theta)+4\varepsilon\,\sin(2\theta)\right)+\right.\\
&\quad \left. 4K\,\sin(\theta)\tanh\left(\gamma\rho\left(\cos(\theta)+c\,\sin(\theta)\right)\right)\right). && (153)
\end{aligned}$$

‡ ‡ ‡

Now let us look at Equation 141 and try to find the first estimate of the region of attraction. From Equation 141

$$\begin{aligned}
\dot{V} &= -0.565\,\rho^2\cos^2(\theta)-0.565\,\rho^3\left(\cos^2(\theta)\sin(\theta)\right)+\\
&\quad \rho\,\cos(\theta)\left(-0.0565+0.1516\,\rho\,\sin(\theta)-0.3569\,\rho^2\sin^2(\theta)\right)+\\
&\quad \rho\,\sin(\theta)\left(0.0435+0.217\,\rho\,\sin(\theta)+0.6085\,\rho^2\sin^2(\theta)\right)+\\
&\quad \rho\,\left(-22.062\,\cos(\theta)+16.95\,\sin(\theta)\right)\tanh\left(50\,\rho\,\cos(\theta)+65\,\rho\,\sin(\theta)\right). && (154)
\end{aligned}$$

By noticing that

$$-0.565\,\rho^2\cos^2(\theta)\le 0,\quad \text{and}\quad \tanh\left(50\,\rho\,\cos(\theta)+65\,\rho\,\sin(\theta)\right)\le 1,$$

Equation 154 becomes

$$\begin{aligned}
\dot{V} &\le \rho\left(-0.0565\,\cos(\theta)+0.0435\,\sin(\theta)-22.062\,\cos(\theta)+16.95\,\sin(\theta)\right)\\
&\quad +\rho^2\left(-0.565+0.1516\,\cos(\theta)\sin(\theta)+0.217\,\sin^2(\theta)\right)\\
&\quad +\rho^3\left(-0.565\,\cos^2(\theta)\sin(\theta)-0.3569\,\cos(\theta)\sin^2(\theta)+0.6085\,\sin^3(\theta)\right) && (155)\\
&\le \rho\left(|-22.1185|\cdot|cos(\theta)|+|16.9935|\cdot|\sin(\theta)|\right)\\
&\quad +\rho^2\left(-0.565+|0.1516|\cdot|\cos(\theta)\sin(\theta)|+|0.217|\cdot\left|\sin^2(\theta)\right|\right)\\
&\quad +\rho^3\left(|-0.565|\cdot\left|\cos^2(\theta)\sin(\theta)\right|+|-0.3569|\cdot\left|\cos(\theta)\sin^2(\theta)\right|+|0.6085|\cdot\left|\sin^3(\theta)\right|\right) && (156)\\
&\le \rho\left(22.1185\times 1+16.9935\times 1\right)+\rho^2\left(-0.565+0.1516\times 0.5+0.217\times 1\right)\\
&\quad +\rho^3\left(0.565\times 0.3849+0.3569\times 0.3849+0.6085\times 1\right) && (157)\\
&\le 31.112\,\rho-0.267\,\rho^2+0.9633\,\rho^3 \quad <0. && (158)
\end{aligned}$$

Equation 158 does not yield any solution $\rho\in\mathcal{R}$ therefore the estimate of the region of attraction has not yet been obtained.

Trying to find the estimate from Equation 135 only leads to another failure since from Equation 135

$$\begin{aligned}\dot{V} &\leq \left(\sqrt{0.0435^2+0.0565^2}+\sqrt{16.95^2+22.062^2}\right)\|x\|_2+\left(\sqrt{0.217^2+0.565^2}+\frac{0.1516}{2}\right)\|x\|_2^2+\\ &\quad\left(0.6085+\frac{\sqrt{0.3569^2+0.565^2}}{2}\right)\|x\|_2^3 \qquad (159)\\ &\leq 0.9428\|x\|_2^3+0.681\|x\|_2^2+27.8928\|x\|_2 \qquad (160)\end{aligned}$$

which also yields no solution $\|x\|_2 \in \mathcal{R}$.
This could be because our assumption that the origin is the equilibrium point was wrong. Looking at Equation 131 at the origin $f_1(x) = \varepsilon$, instead of zero. The Lyapunov function used above was obtained from solving the Lyapunov equation using the matrix A obtained with the above assumption. So if the assumption was not sound, A loses its validity.
Let us look more carefully at the response of the system. Plot Figure 79 with the Initial Condition Set 6 which is similar to the Initial Condition Set 1 often refered to before but with a smaller scale. Figure 79 shows that the system has a stable focus at approximately $(0.5, -1)$.

Initial Condition Set 6 *The set of n points described by*

```
n = 0
for i = -0.5 to 0.5 step 0.02
     n = n+1, calculate the n_th initial values as x1(0) = i, x2(0) = i;
     n = n+1, calculate the n_th initial values as x1(0) = i, x2(0) = -i;
endfor
```

Let us do the simulation again to investigate the behaviour of the system inside the limit cycle by shifting the Initial Condition Set 6 downward along the x_2-axis and obtain the Initial Condision Set 7. Use the new initial condition set to simulate Figure 80. From the response shown the trajectory starting from inside the limit cycle moves outward toward it.

Initial Condition Set 7 *The set of n points described by*

```
n = 0
for i = -0.5 to 0.5 step 0.02
     n = n+1, calculate the n_th initial values as x1(0) = i, x2(0) = i-1;
     n = n+1, calculate the n_th initial values as x1(0) = i, x2(0) = -i-1;
endfor
```

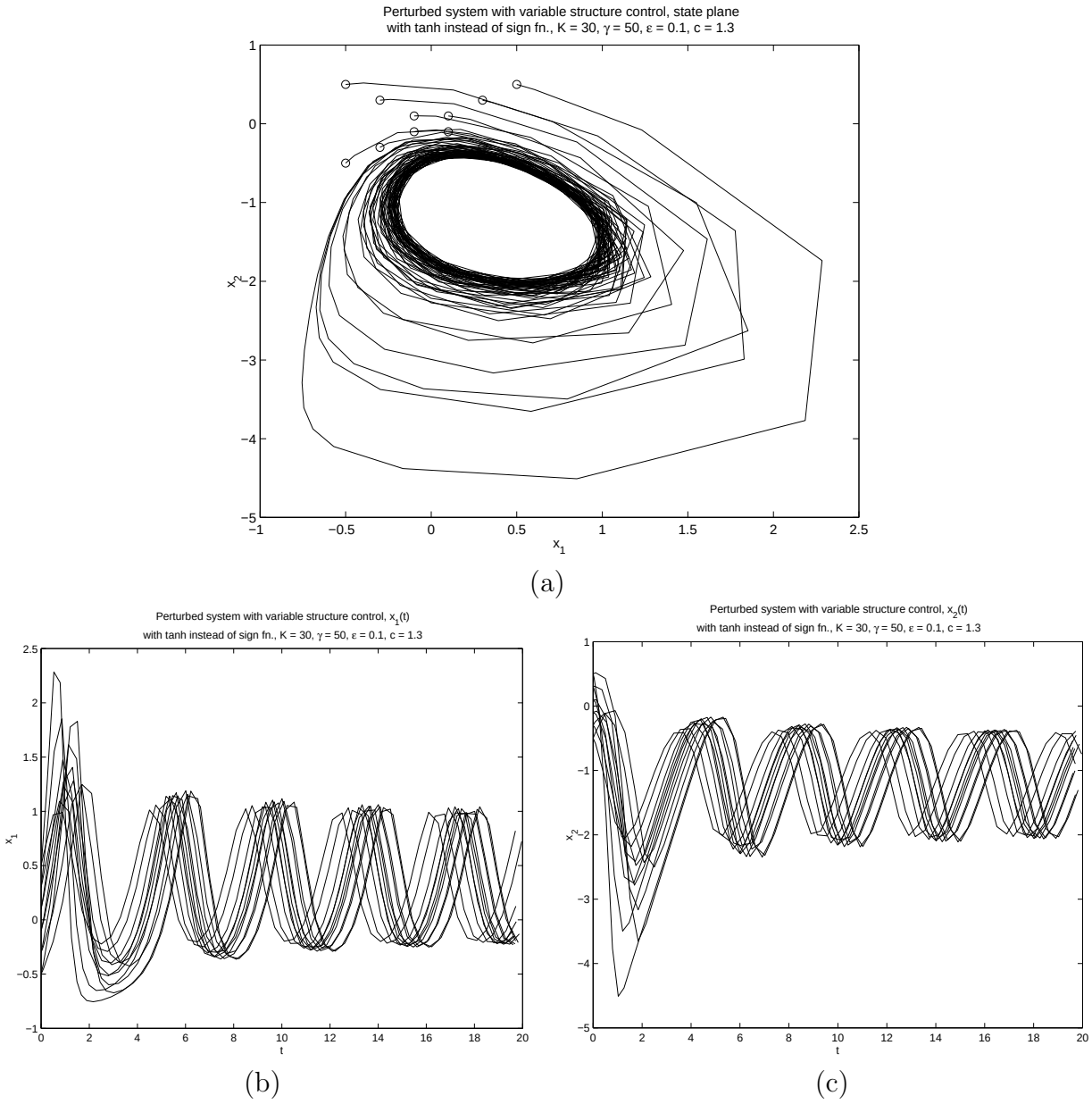

Perturbed system with variable structure control, state plane
with tanh instead of sign fn., K = 30, γ = 50, ε = 0.1, c = 1.3
x1
x2
(a)
Perturbed system with variable structure control, x1(t)
with tanh instead of sign fn., K = 30, γ = 50, ε = 0.1, c = 1.3
t
(b)
Perturbed system with variable structure control, x2(t)
with tanh instead of sign fn., K = 30, γ = 50, ε = 0.1, c = 1.3
t
(c)

Figure 79: *Tanh variable structure controlled response (a) state plane, (b) $x_1(t)$, (c) $x_2(t)$*

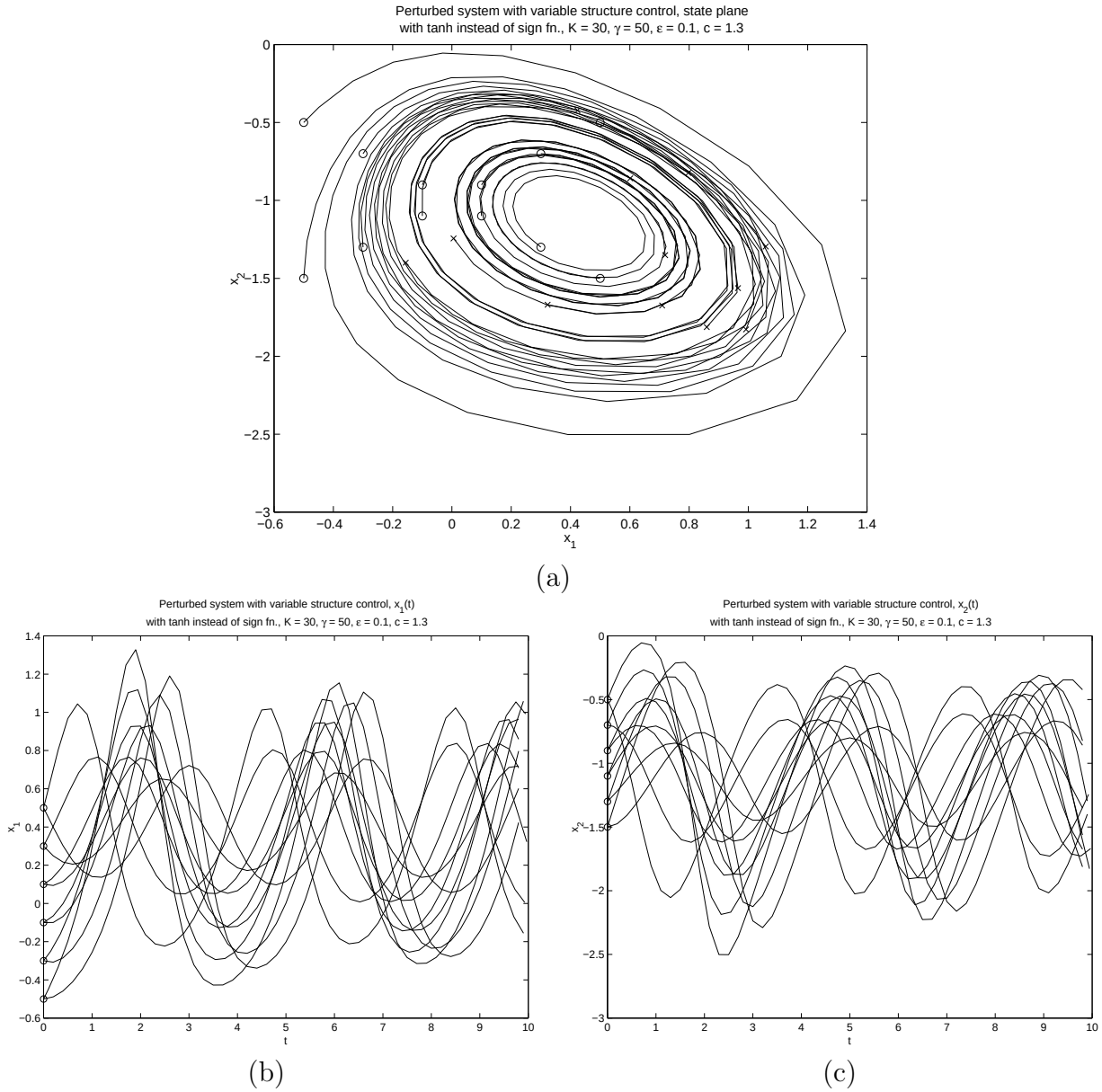

Perturbed system with variable structure control, state plane
with tanh instead of sign fn., K = 30, γ = 50, ε = 0.1, c = 1.3
x1
x2
(a)
Perturbed system with variable structure control, x1(t)
with tanh instead of sign fn., K = 30, γ = 50, ε = 0.1, c = 1.3
t
(b)
Perturbed system with variable structure control, x2(t)
with tanh instead of sign fn., K = 30, γ = 50, ε = 0.1, c = 1.3
t
(c)

Figure 80: *Tanh variable structure controlled response (a) state plane, (b) $x_1(t)$, (c) $x_2(t)$ inside the limit cycle*

With the light of Figure 79 and Figure 80 one could use, say a radius 0.5 circle centered at $(0.5, -1)$, as an ad hoc first estimate of the region of attraction boundary, and then proceed on with the trajectory-reversing method to find better estimates.
But let us try an approach which is a little bit more systematic. Let us unperturbed the system by setting $\varepsilon = 0$, then use the matrix A obtained to find the region of attraction. When $\varepsilon = 0$ Equation 127 and 128 yield the origin as an equilibrium point. The Jacobian matrix becomes

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -1 + x_2 & 1 + x_1 \\ -1 - 2x_1 - c\gamma K \operatorname{sech}(\gamma(cx_1 + x_2)) & -\gamma K \operatorname{sech}(\gamma(cx_1 + x_2)) \end{bmatrix} \tag{161}$$

and at the equilibrium point becomes

$$A = \begin{bmatrix} -1 & 1 \\ -1 - c\gamma K & -\gamma K \end{bmatrix}. \tag{162}$$

With the values of parameters used above

$$A = \begin{bmatrix} -1 & 1 \\ -1951 & -1500 \end{bmatrix}. \tag{163}$$

The eigenvalues of this matrix are

$$\lambda_1 = -2.3, \qquad \lambda_2 = -1498.7.$$

Solve the Lyapunov equation (Equation 102) for P using A in Equation 163 leads to

$$P = \begin{bmatrix} 0.2175 & -0.2825 \\ -0.2825 & 0.3677 \end{bmatrix} \tag{164}$$

hence the Lyapunov function can be found from Equation 130 to be

$$V = 0.2175\, x_1^2 - 0.565\, x_1 x_2 + 0.3677\, x_2^2 \tag{165}$$

which the derivative is

$$\begin{aligned} \dot{V} &= 0.13\, x_1^2 + 0.565\, x_1^3 + 0.2646\, x_1 x_2 - 0.3004\, x_1^2 x_2 - 0.565\, x_2^2 - 0.565\, x_1 x_2^2 + \\ &\quad 16.95\, x_1 \tanh(50(1.3\, x_1 + x_2)) - 22.062\, x_2 \tanh(50(1.3\, x_1 + x_2)) \qquad (166) \\ &\leq \left(0.13\, x_1^2 + 0.2646\, x_1 x_2 - 0.565\, x_2^2\right) + \left(0.565\, x_1^3 - 0.3004\, x_1^2 x_2 - \right. \\ &\quad \left. 0.565\, x_1 x_2^2\right) + (16.95\, x_1 - 22.062\, x_2) \tanh(50(1.3\, x_1 + x_2)) \\ &\leq \sqrt{16.95^2 + 22.062^2}\|x\|_2 + (0.13 + 0.2646 - 0.565)\|x\|_2^2 + \\ &\quad \left(0.565 + \frac{\sqrt{0.3004^2 + 0.565^2}}{2}\right)\|x\|_2^3. \qquad (167) \end{aligned}$$

Equation 167 does not yield a solution $\|x\|_2 \in \mathcal{R}$ either.
Since we could not find a first estimate of the region of attraction, then the trajectory-reversing method can not be applied. But I will try using the trajectory-reversing method by starting from Figure 79 and 80 which show a limit cycle close to

$$(x_1 - 0.5)^2 + (x_2 + 1)^2 = 0.5^2. \tag{168}$$

Starting from 20 points on the contour of Equation 168 using the convention given in page 'points' simulate the inverse system in the form given by Equation 111. The Equation 127 and 128 with the set of parameter values used above become

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2\left(1+x_1\right) - 0.1\left(1+x_1\right)^2 && (169)\\ \dot{x}_2 &= x_1\left(1+x_1\right) + K\,\tanh\left(\gamma\left(cx_1+x_2\right)\right). && (170) \end{aligned}$$

And the plot upto the 49^{th} time step, ie

$$0 \le t_i \le 0.01 \text{ seconds}, \qquad i = 1, 2, \ldots, 49,$$

is shown in Figure 81 (a) while in Figure 81 (b) the time steps were

$$0 \le t_i \le 0.5 \text{ seconds}, \qquad i = 1, 2, \ldots, 51.$$

Figure 81 (c) and (d) are both closed-ups for Figure 81 (b).
The figure shows a trajectory which seems to be on a fixed course, which could be a characteristic of a system subjected to a control input. Eventhough my approach in obtaining Figure 81 comes from looking at a phase plane response (therefore ad hoc) instead of at the Lyapunov function (which would have been more mathematical, ie more intrigue) I do believe that this approach works with a system if it is continuous.

The following is only to fill in the blank space of this and the next pages.
Three inequalities often used in considering the upperbound of a Lyapunov function are

$$\begin{aligned} |x_1| &\le \|x\|_2, && (171)\\ |x_1x_2| &\le \frac{1}{2}\|x\|_2^2, \quad \text{and} && (172)\\ |ax_1 \pm bx_2| &\le \sqrt{a^2+b^2}\|x\|_2, && (173) \end{aligned}$$

where $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$.
About a hyperbolic tangent function, written in exponential form it is

$$\tanh x = \frac{-e^{-x} + e^x}{e^{-x} + e^x}. \tag{174}$$

Its taylor series upto the 10^{th}-order is

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \mathcal{O}^{11}(x). \tag{175}$$

And the taylor series of

$$f = \tanh\left(\gamma\left(cx_1+x_2\right)\right),$$

where $\gamma, c \in \mathcal{R}$ and upto the 5^{th}-order is

$$\begin{aligned} f &= \gamma x_2 - \frac{\gamma^3 x_2^3}{3} + \frac{2\gamma^5 x_2^5}{15} + \mathcal{O}^6\left(x_2\right) + \\ &\quad \left(c\gamma - c\gamma^3 x_2^2 + \frac{2c\gamma^5 x_2^4}{3} + \mathcal{O}^6\left(x_2\right)\right) x_1 + \end{aligned}$$

$$\left(-c^2\gamma^3 x_2 + \frac{4c^2\gamma^5 x_2^3}{3} - \frac{17c^2\gamma^7 x_2^5}{15} + \mathcal{O}^6(x_2)\right) x_1^2 +$$
$$\left(\frac{-c^3\gamma^3}{3} + \frac{4c^3\gamma^5 x_2^2}{3} - \frac{17c^3\gamma^7 x_2^4}{9} + \mathcal{O}^6(x_2)\right) x_1^3 +$$
$$\left(\frac{2c^4\gamma^5 x_2}{3} - \frac{17c^4\gamma^7 x_2^3}{9} + \frac{124c^4\gamma^9 x_2^5}{45} + \mathcal{O}^6(x_2)\right) x_1^4 +$$
$$\left(\frac{2c^5\gamma^5}{15} - \frac{17c^5\gamma^7 x_2^2}{15} + \frac{124c^5\gamma^9 x_2^4}{45} + \mathcal{O}^6(x_2)\right) x_1^5 + \mathcal{O}^6(x_1) \tag{176}$$

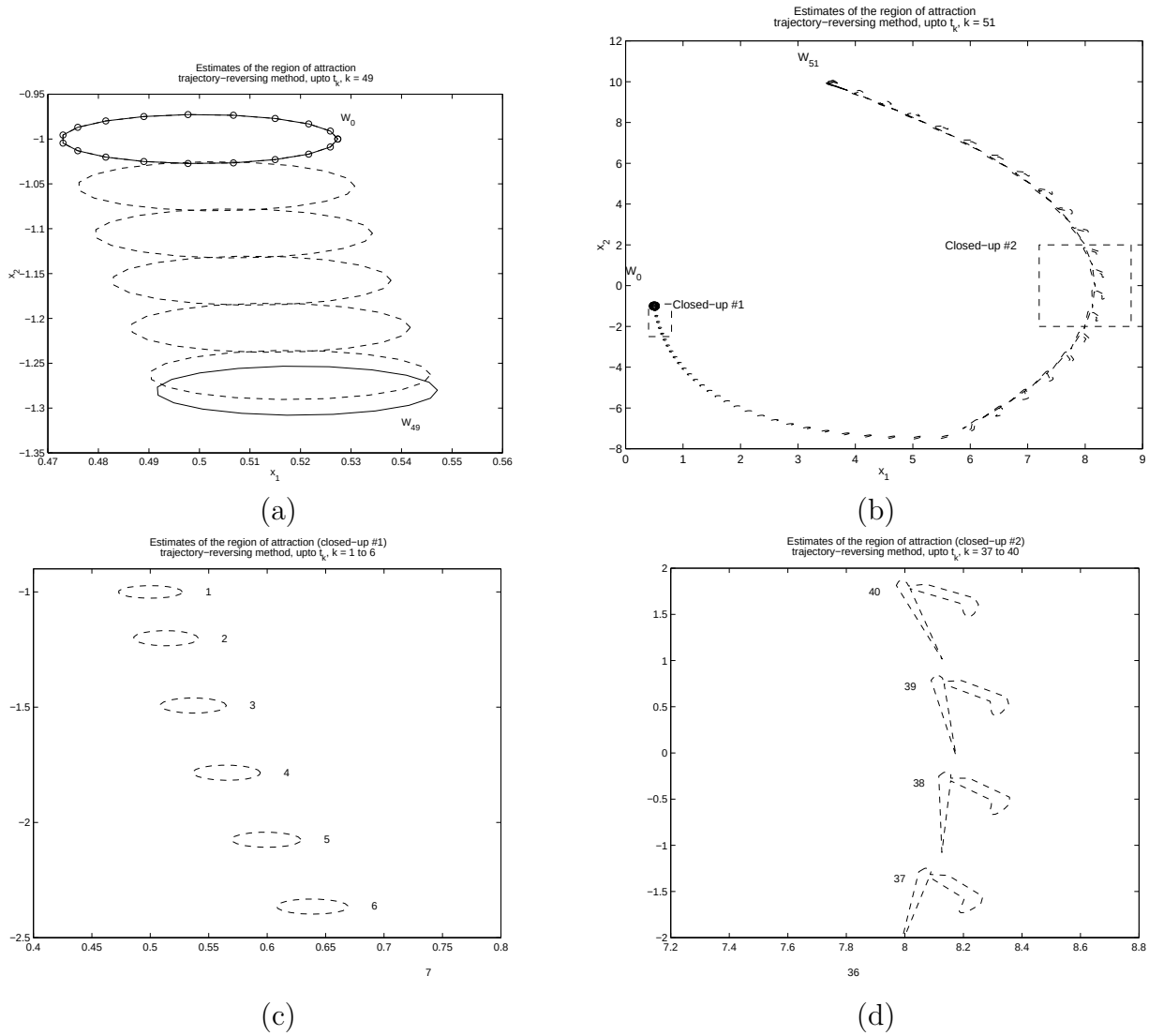


Figure 81: *Region of attraction (a) $t \leq 0.01$ sec, (b) $t \leq 0.5$ sec, (c) and (d) closed-up's of (b)*

A 4^{th} order singularly perturbed system

This section is the addition and correction to the result in Technical Report #1 [Tiy98a] (see also [Tiy98b]). Consider the same system to a system studied in Technical Report #1,

$$\dot{x} = Ax + Bu \tag{177}$$
$$y = Cx + Du \tag{178}$$

with

$$A = \begin{bmatrix} -8 & -5 & 1 & 1.25 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -11 & -2.5 \\ 0 & 0 & 4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 2.82843 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.82843 & 1.59099 & 0.70711 & 0.88388 \end{bmatrix}$$

and

$$D = 0.$$

Response of the system without an input

Let $\varepsilon = 1$

Incorporation of ε

The system equations becomes

$$\dot{x} = E^{-1}Ax + E^{-1}Bu \tag{179}$$
$$y = Cx + Du. \tag{180}$$

when

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}$$

is incorporated into the system equations. Notice that here $E^{-1}B = B$ and $E^{-1}A$ only differs from A in the 4^{th} row which has only got one element (column 3.) Also $CE^{-1}B = 2$ regardless of the values of ε.

Some knowledge about the new system

The equilibrium point still remains at the origin but the eigenvalues are affected by the ε introduced and are

$$-4 \pm \jmath 2, \quad \text{and} \quad -\frac{11}{2} \pm \frac{1}{2}\sqrt{121 - 40\varepsilon}$$

Phase plane plots of the new system with different ε's

The following figures are the plots of the system without a control input. When $\varepsilon = 1$ the state plane plot were shown in Figure 82. The initial conditions were taken as

Initial Condition Set 8 *is a set of points described by*

```
for i=-5,5
  for j=-5,-3,-1,1,3,5
    for (p,q)=(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)
      x_p(0) = i
      x_q(0) = j
      x_k(0) = 0, for all k which are not equal to p or q
    endfor
endfor
endfor
```

Figure 83 shows the state planes obtained when $\varepsilon = 0.01$.
Figure 84 shows the state planes obtained when $\varepsilon = 10^{-4}$.

State variables plot

The following are the time domain plots of the state variables for the values of ε's previously used. Figure 85 shows the state variables and state metamorphosis when $\varepsilon = 1$. The phase metamorphosis diagram was simulated upto $t = 6$ seconds. Shown on the diagram, the term t_{final} is used to refer to the time when all the state variables converge to the origin to within ± 0.001 tolerance. Notice that $t_{\text{final}} < 6$ seconds The initial conditions used were

Initial Condition Set 9 *is a set of point described by*

```
for i = -3, 3
  for j = -3, 3
    for k = -3, 3
      for m = -3, 3
        x_1(0) = i, x_2(0) = j, x_3(0) = k, x_4(0) = m
      endfor
    endfor
  endfor
endfor
```

Figure 86 shows the state variables and state metamorphosis when $\varepsilon = 0.01$.
Figure 87 shows the state variables and the metamorphosis of states when $\varepsilon = 10^{-4}$.
Figure 88 shows the result simulated for $\varepsilon = 0.1$ using Initial Condition Set 9 as the intial conditions.

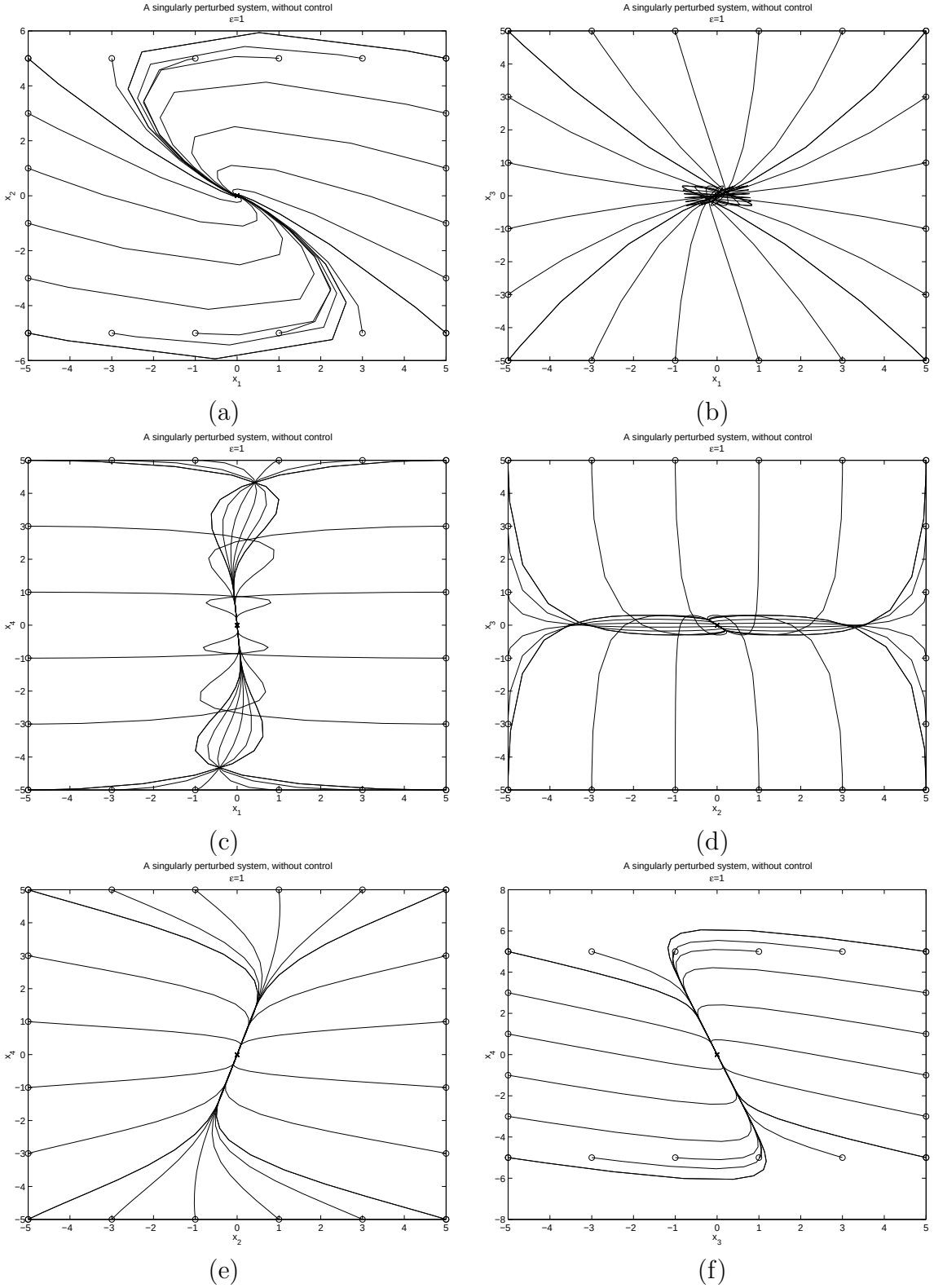


Figure 82: *State diagrams of a system with* $\varepsilon = 1$

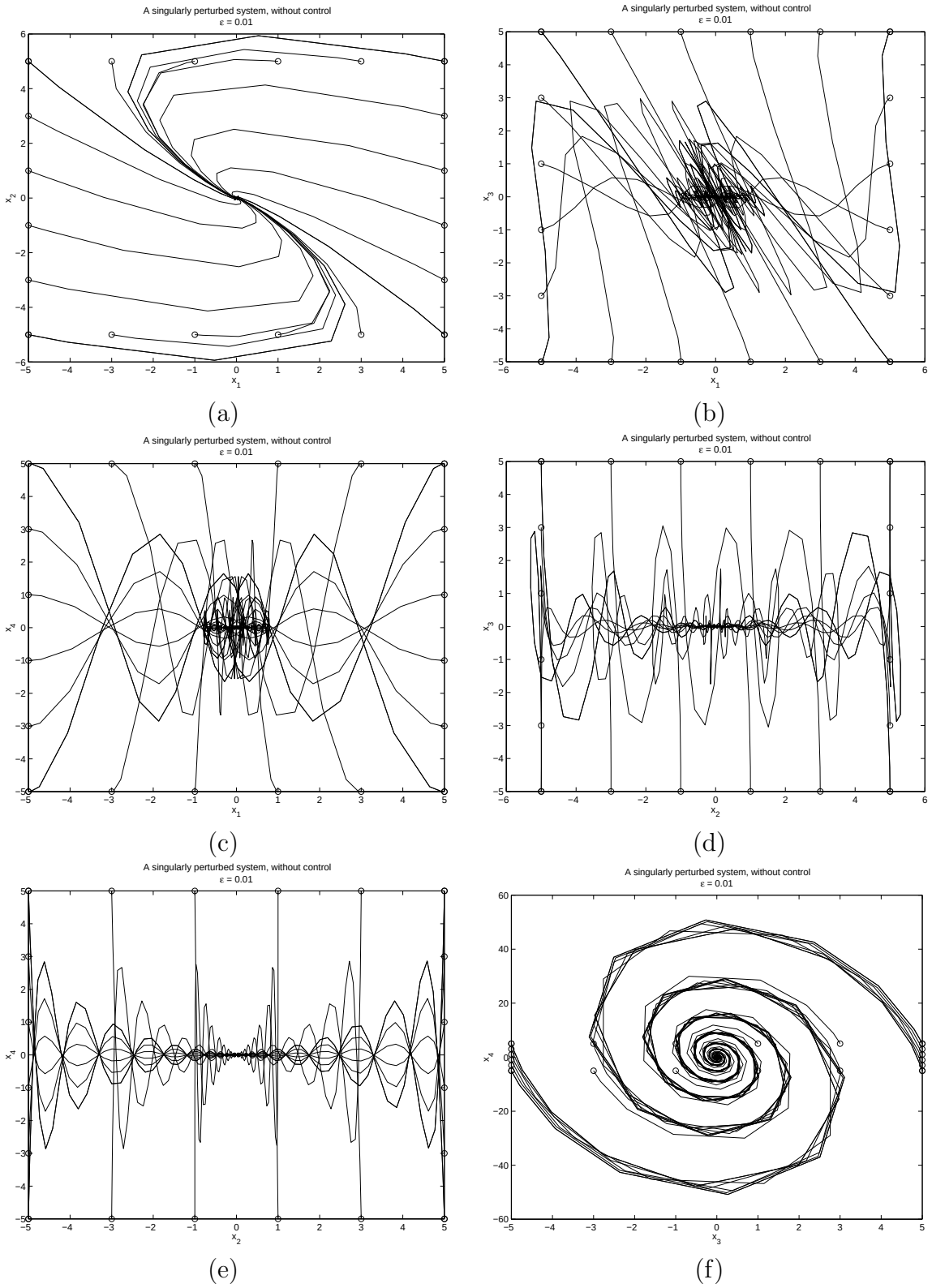


Figure 83: *State diagrams of a system with* $\varepsilon = 0.01$

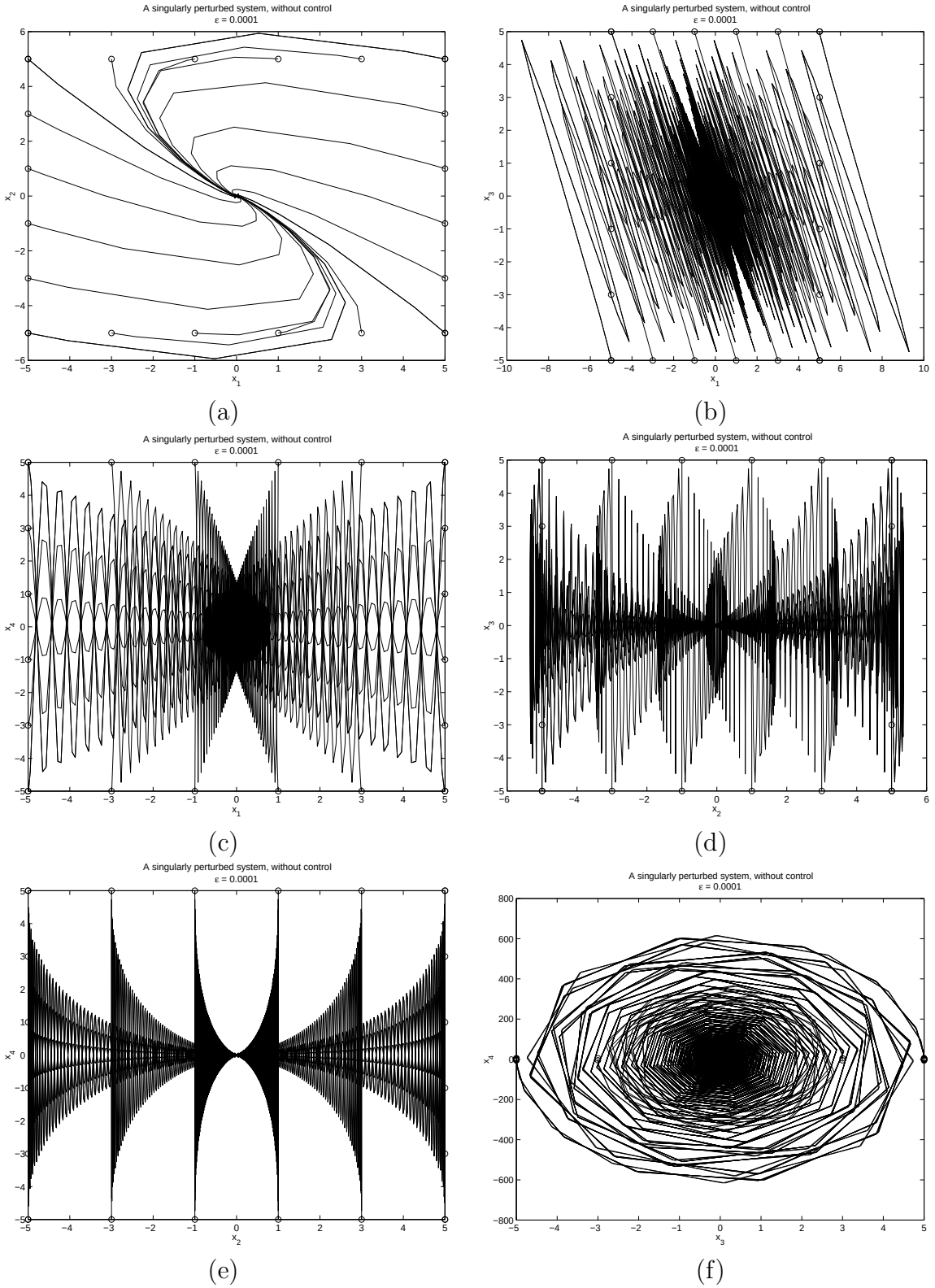


Figure 84: *State diagrams of a system with* $\varepsilon = 10^{-4}$

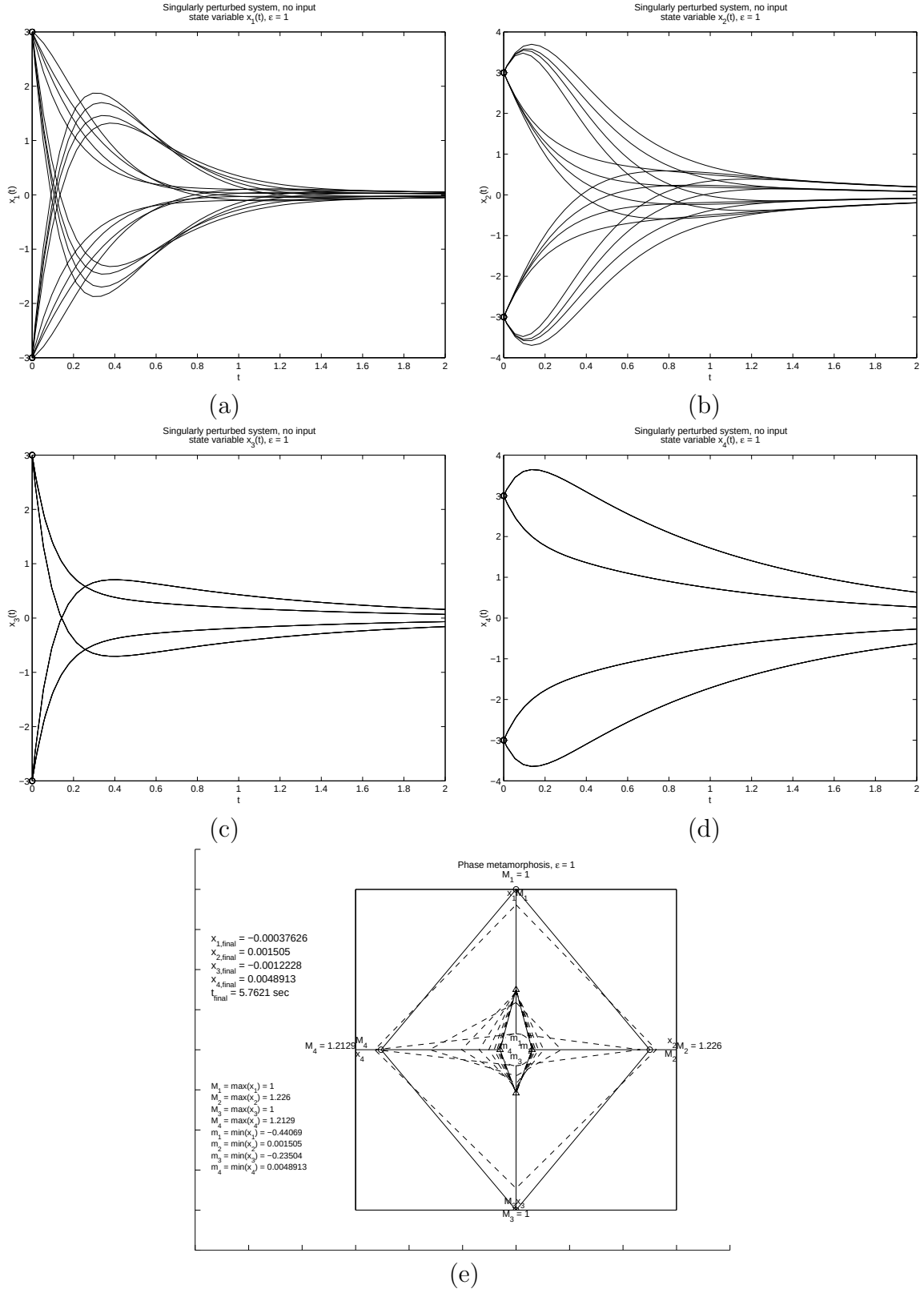


Figure 85: *(a) to (d) state variables of a system with $\varepsilon = 1$, (e) phase metamorphosis diagram*

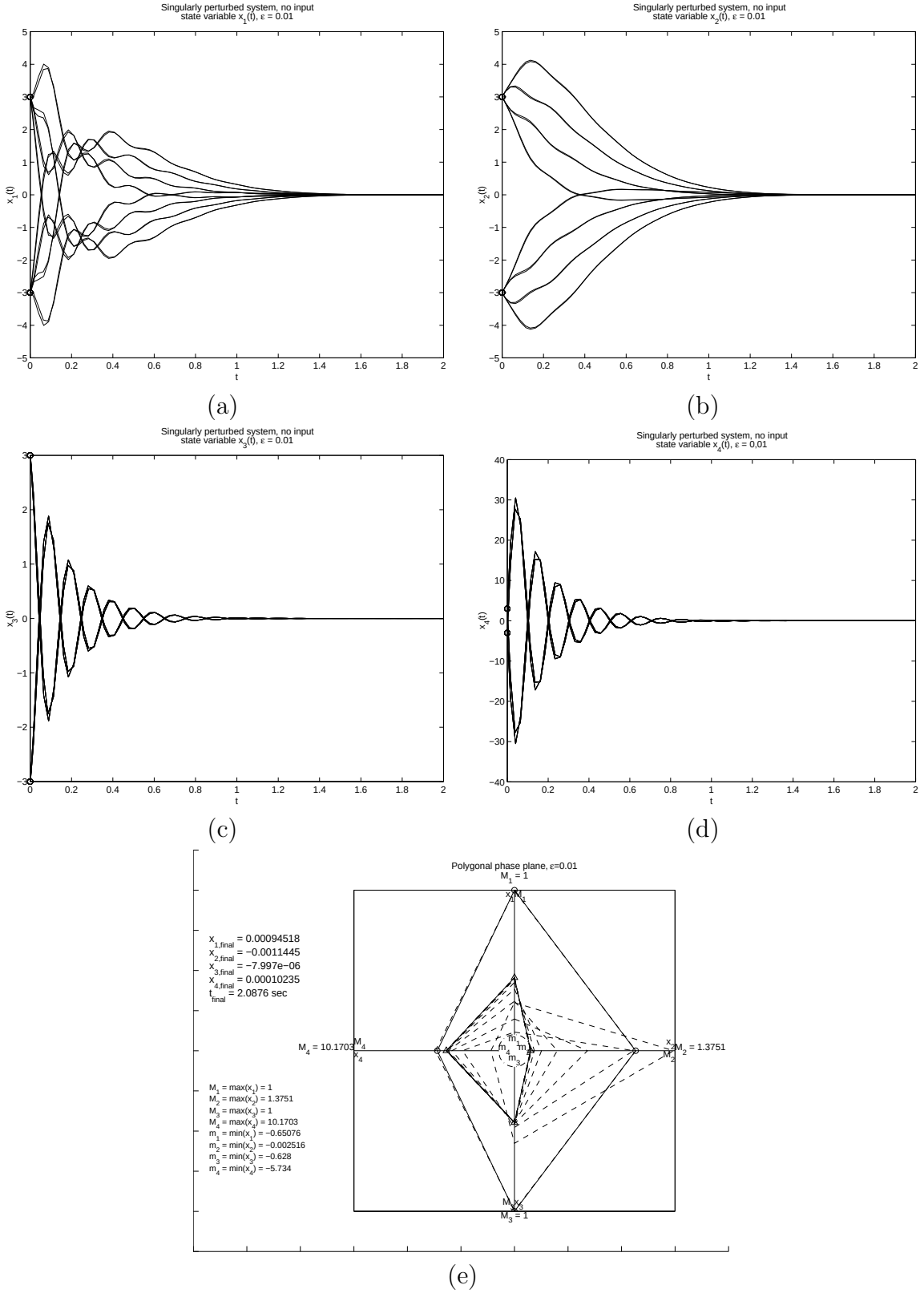


Figure 86: *(a) to (d) state variables of a system with* $\varepsilon = 0.01$*, (e) polygonal phase metamorphosis diagram*

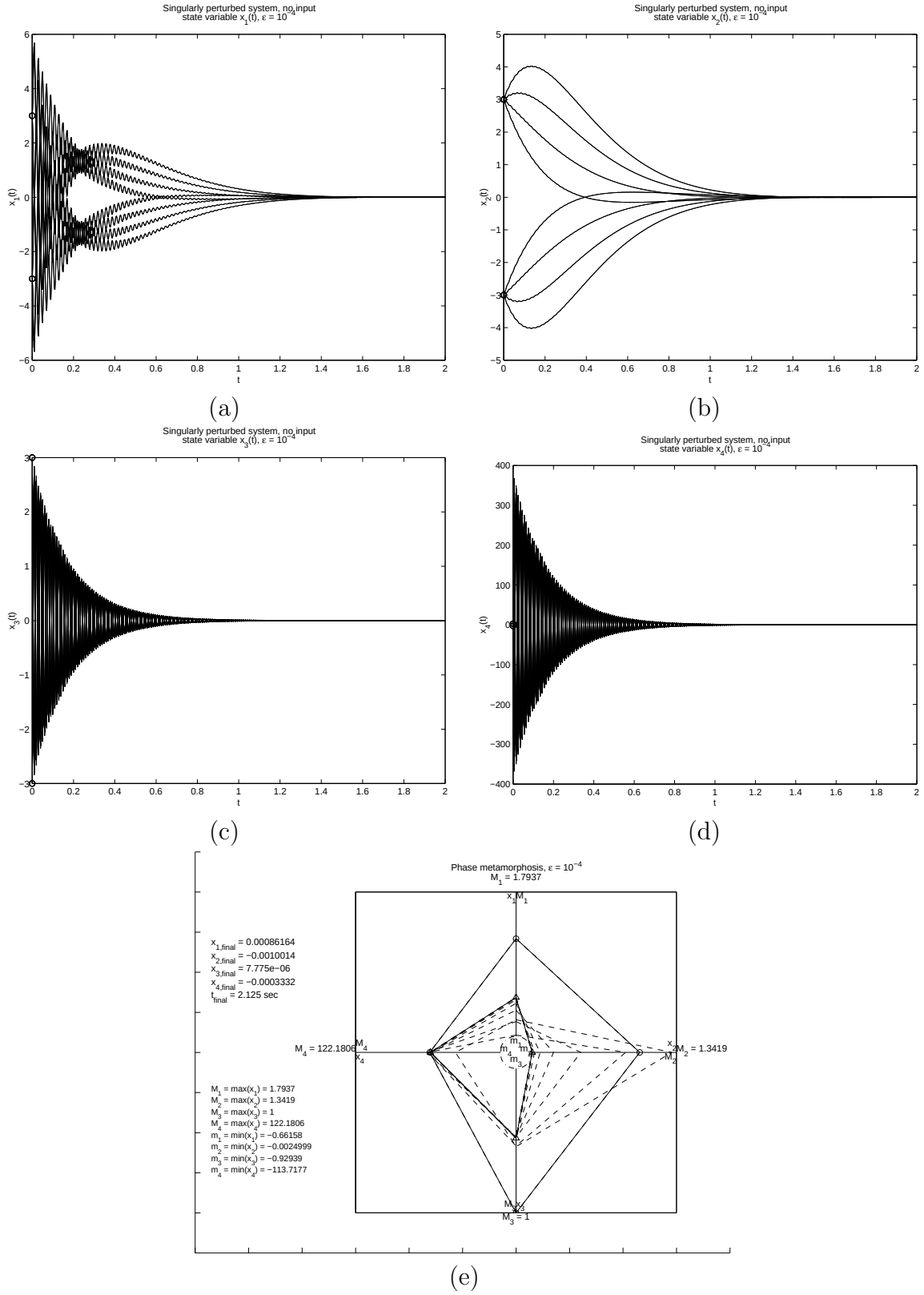


Figure 87: *(a) to (d) state variables of a system with* $\varepsilon = 10^{-4}$*, (e) polygonal phase metamorphosis diagram*

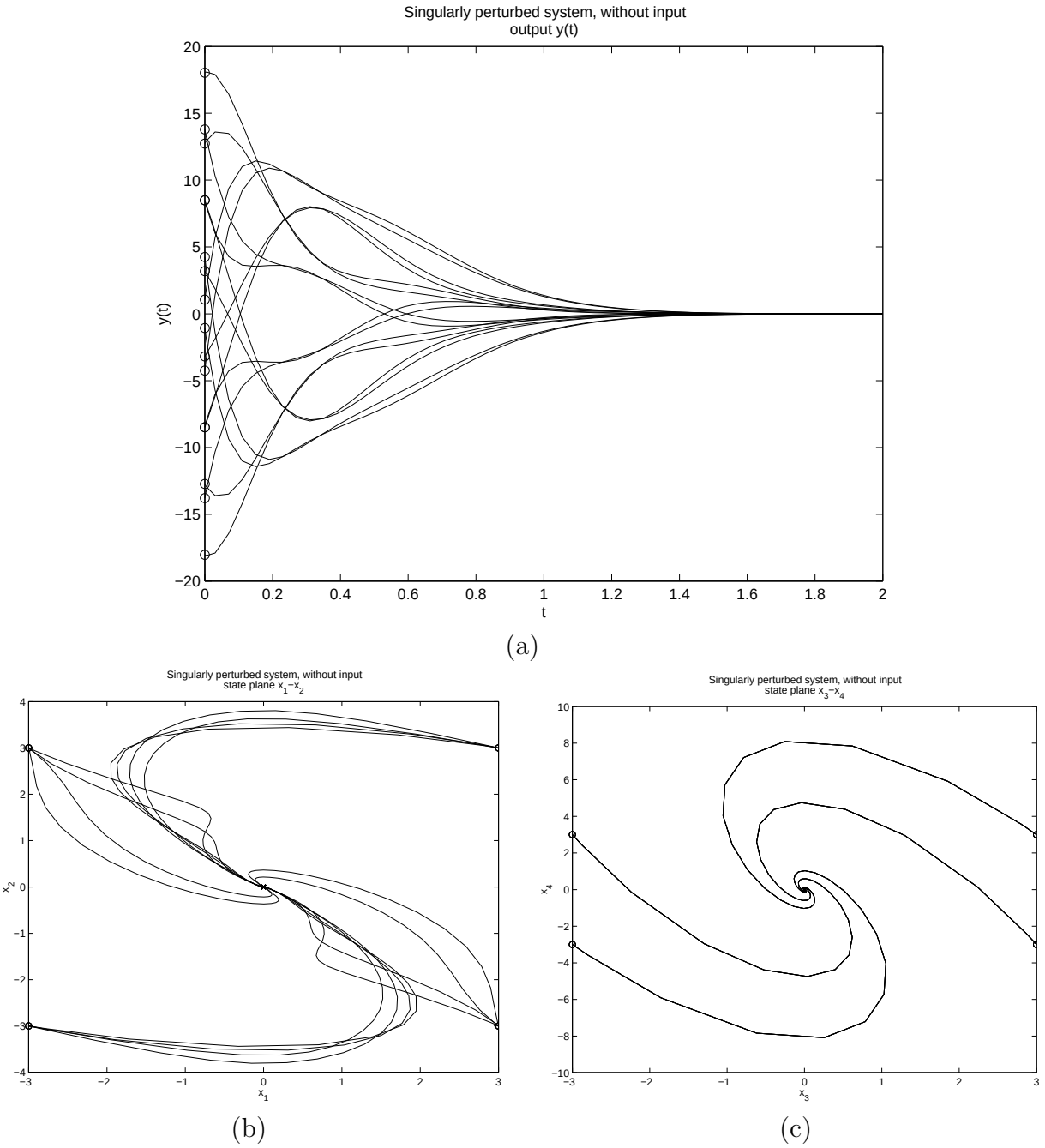


Figure 88: *Output $y(t)$ (a) and state diagrams ((b) and (c)) of the system with $\varepsilon = 0.1$, no input*

Response of the system with a VS feedback input

Design a controller with an input signal

$$u = Cx + \mathrm{sgn}(\mathrm{CE}^{-1}\mathrm{Ax} + \mathrm{CE}^{-1}\mathrm{Bu}), \tag{181}$$

where

$$\mathrm{sgn(f)} = \begin{cases} 1 & , f > 0 \\ 0 & , f = 0 \\ -1 & , f < 0. \end{cases}$$

Equation 181 could not be simulated because algebraic loops were encountered (according to MATLAB's error messages.) Therefore modified equation used instead was

$$u = Cx + \mathrm{sgn}(\mathrm{CE}^{-1}\mathrm{Ax}). \tag{182}$$

When $\varepsilon = 1$ the result can be shown as in Figure 89. Figure 89 is quite similar to Figure 82 where the system was without any control and with the same $\varepsilon = 1$. Here the initial conditions used were the same as those used for Figure 82.

The next picture, Figure 90, shows the result of a simulation when $\varepsilon = 0.01$. Figure 90 should be compared with Figure 83 which is the case with the same value of ε ($\varepsilon = 0.01$), but which has no control input. Both Figure 90 and Figure 83 has the same set of initial conditions.

For the case where $\varepsilon = 10^{-4}$ the result is show in Figure 91. Figure 91 can be compared with Figure 84 to see the effect which the feedback applied has on the system response. The initial conditions used in Figure 91 were the same as those used for obtaining Figure 84.

The plots of the state variables when a variable structure feedback is applied to the system with $\varepsilon = 1$ are shown in Figure 92 (together with a phase metamorphosis.) The phase metamorphosis diagram shown in Figure 92 (e) has $(1, 1, 1, 1)$ as the initial condition.

When $\varepsilon = 0.01$ the state variables plot in time domain are shown in Figure 93.

The state variables plot in time domain when $\varepsilon = 10^{-4}$ are shown in Figure 94. In Figure 92, 93, and 94 a hyperbolic tangent function was used instead of a sign function, so that instead of Equation 181 the input is

$$u = Cx + \tanh\left(70\left(CE^{-1}Ax + CE^{-1}Bu\right)\right). \tag{183}$$

Then for comparison with Figure 88, Figure 95 was plotted.

Discussion

The two poles which are not affected by the value of ε (the ones located at $-4 \pm 2\imath$.) have the natural frequency $\omega_n = 4.4721$ and the damping factor 0.8944. (The natural frequency of a linear time invariant system pole is

$$\omega_n = \frac{|\log(\lambda)|}{t_s},$$

where t_s is the sampling time. The damping factor is

$$\zeta = -\cos(\angle(\log\lambda))),$$

λ being the eigenvalue.

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[Tiy98b] Kittisak Tiyapan. Variable structure control for a singularly perturbed system. In *Advance Technology in Automatic Control Systems*, pages Fu–C1–8. Furuta Laboratory, Tokyo Institute of Technology, September 1998.

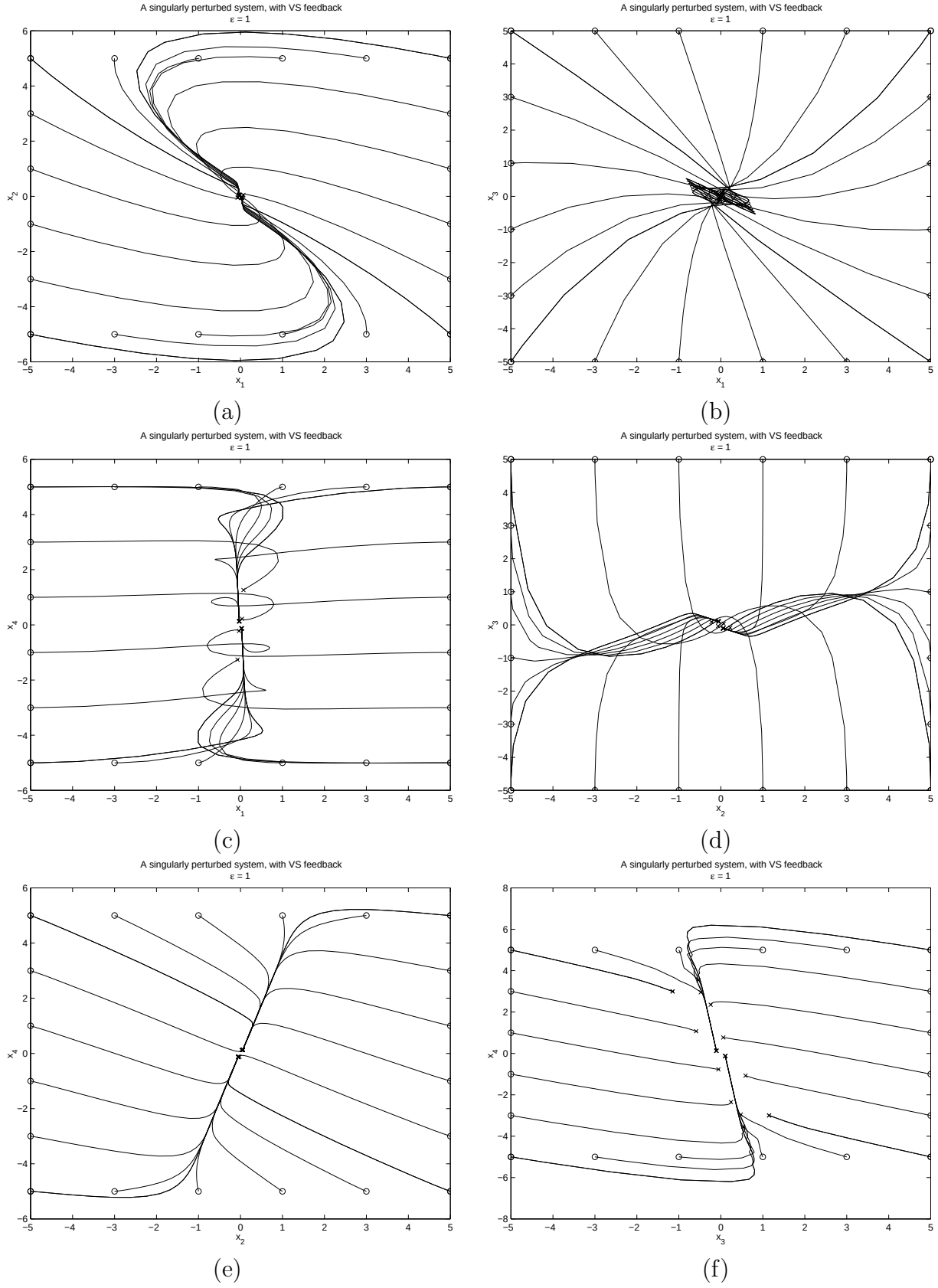

Figure 89: *State diagrams of a feedback system with* $\varepsilon = 1$

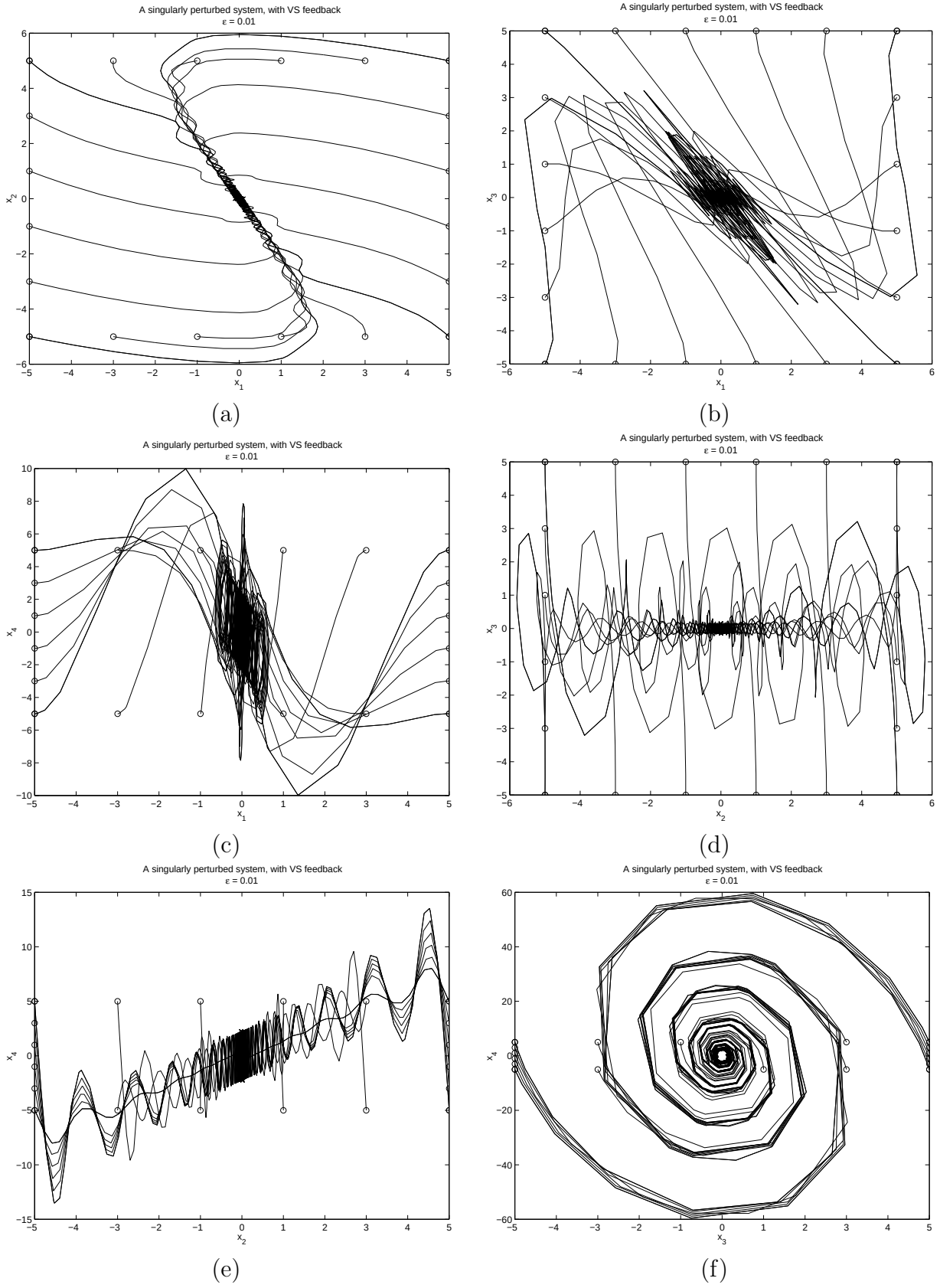

Figure 90: *State diagrams of a feedback system with* $\varepsilon = 0.01$

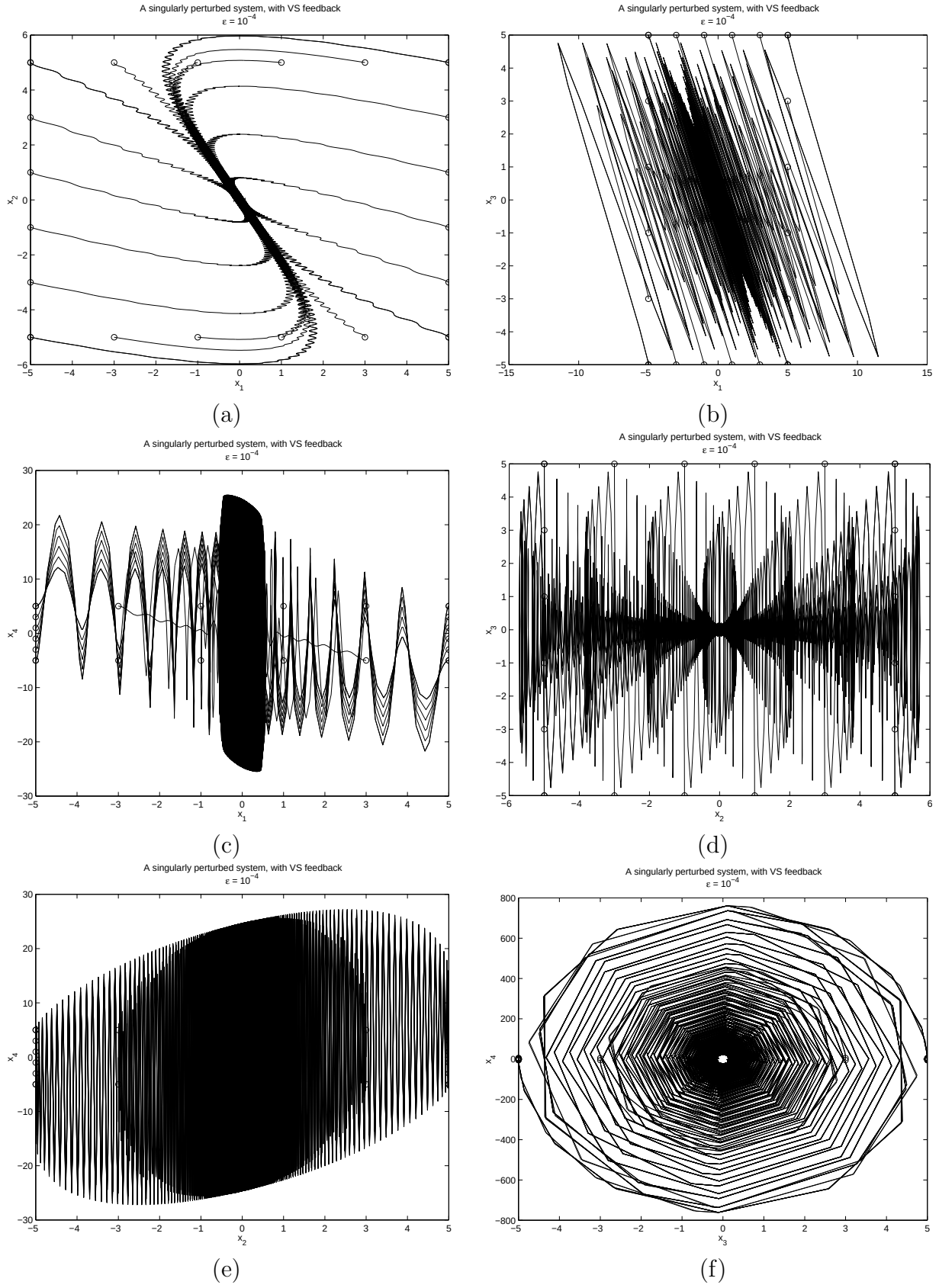

Figure 91: *State diagrams of a feedback system with* $\varepsilon = 10^{-4}$

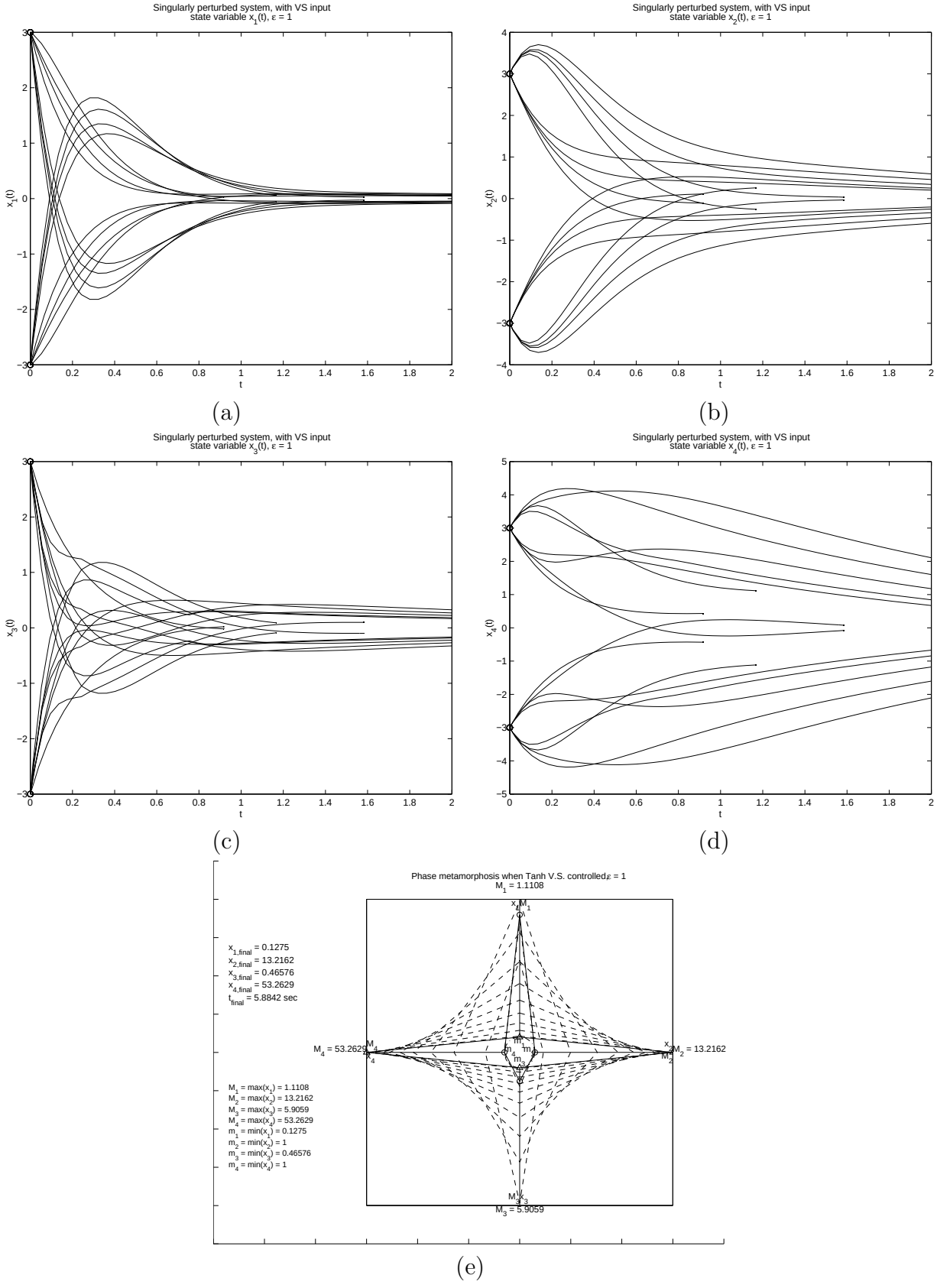

Figure 92: *(a) and (b) state variables of the system with V.S. feedback, $\varepsilon = 1$, (e) metamorphosis when controlled with Tanh V.S.*

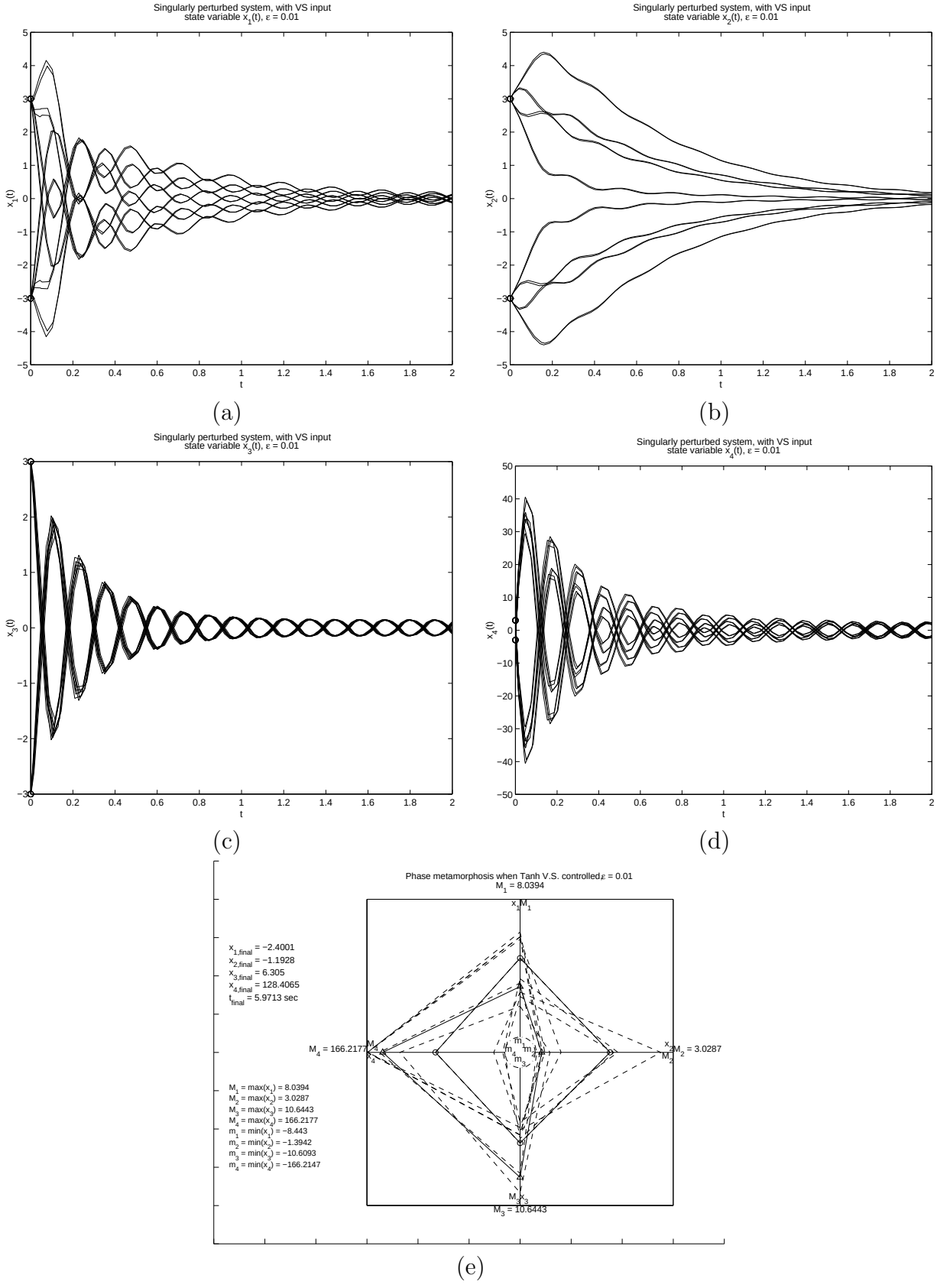

Figure 93: *(a) and (b) state variables of the system with V.S. feedback,* $\varepsilon = 0.01$*, (e) metamorphosis when controlled with Tanh V.S.*

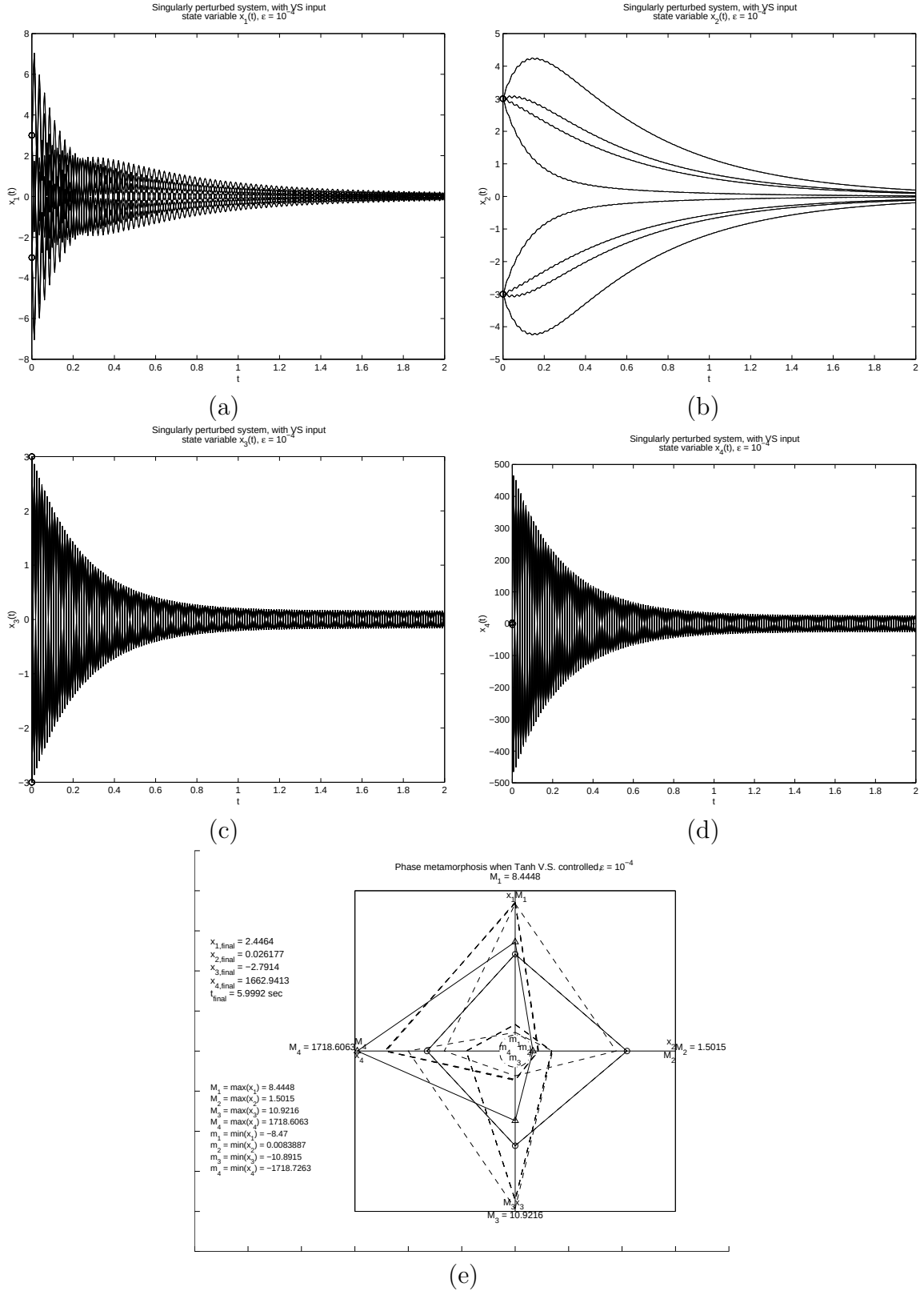

Figure 94: *(a) and (b) state variables of the system with V.S. feedback, $\varepsilon = 10^{-4}$, (e) metamorphosis when controlled with Tanh V.S.*

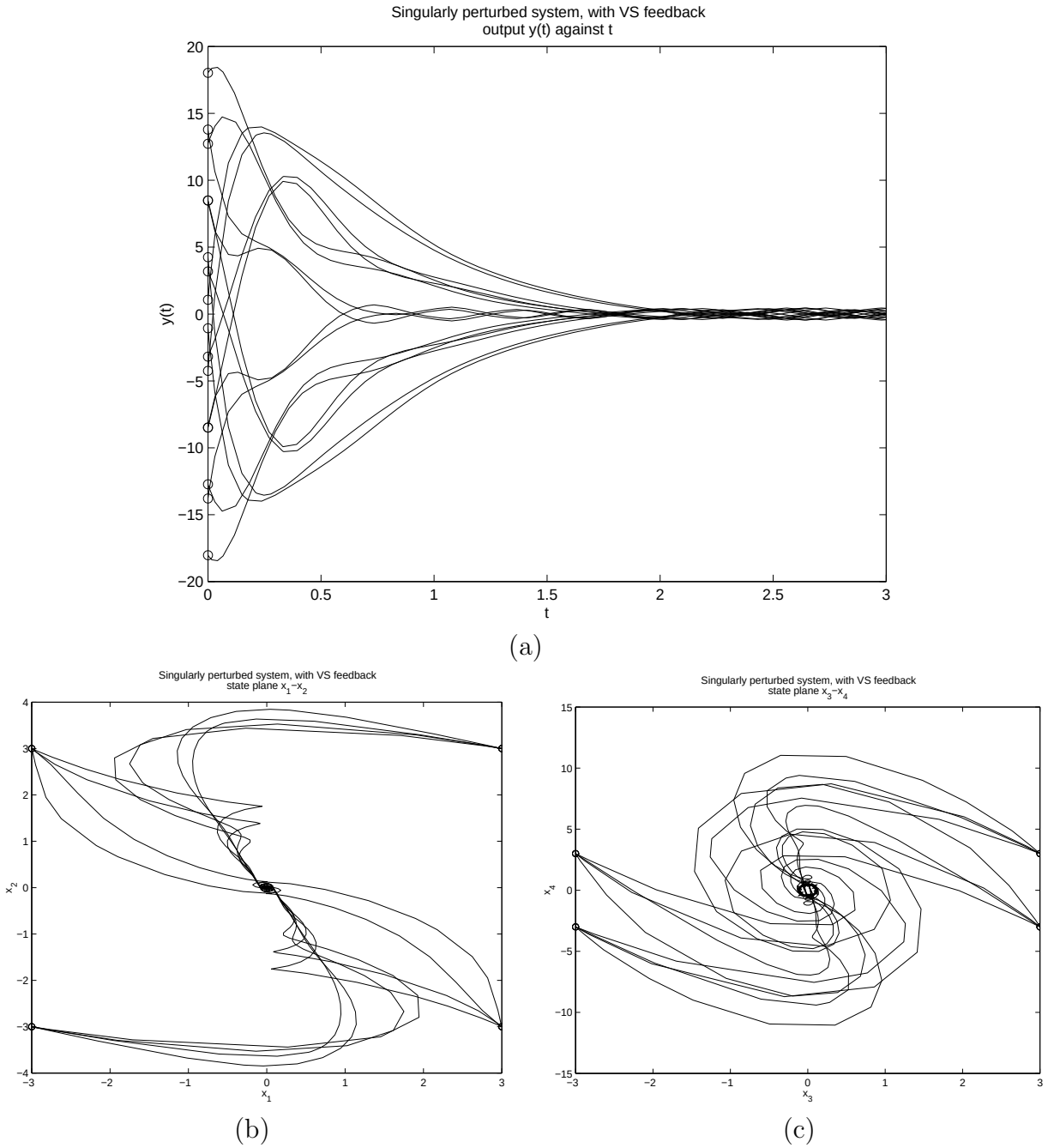

Figure 95: *Output $y(t)$ (a) and state diagrams ((b) and (c)) of the feedback system with $\varepsilon = 0.1$*